

A PROBLEM OF KOLLÁR AND LARSEN ON FINITE LINEAR GROUPS AND CREPANT RESOLUTIONS

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ABSTRACT. The notion of **age** of elements of complex linear groups was introduced by M. Reid and is of importance in algebraic geometry, in particular in the study of crepant resolutions and of quotients of Calabi-Yau varieties. In this paper, we solve a problem raised by J. Kollár and M. Larsen on the structure of finite irreducible linear groups generated by elements of $\text{age} \leq 1$. More generally, we bound the dimension of finite irreducible linear groups generated by elements of bounded deviation. As a consequence of our main results, we derive some properties of symmetric spaces $GU_d(\mathbb{C})/G$ having shortest closed geodesics of bounded length, and of quotients \mathbb{C}^d/G having a crepant resolution.

1. INTRODUCTION

Let $V = \mathbb{C}^d$ be a d -dimensional complex space and let $G < GL(V)$ be a finite subgroup. A classical theme in group theory and representation theory, going back at least to work of H. Blichfeldt on primitive linear groups, and work of G. C. Shephard and J. A. Todd [ST] on complex reflection groups, is to characterize G under various conditions that force G to contain non-identity elements which are “close” to the identity transformation on V . Recall that a *complex reflection group* (c.r.g. for short) is a subgroup of $GU(V)$ that is generated by a set of complex (pseudo)reflections.

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The complex reflection groups can be arguably said to be one of the most ubiquitous objects in modern mathematics.

Recently, motivated by potential applications in algebraic geometry, string theory, mirror symmetry, and quantum cohomology, J. Kollár and M. Larsen [KL] have raised the problem of studying linear groups containing elements of bounded (or small) *deviation*, where the deviation is defined in a certain way to measure the “closeness” of group elements to the identity transformation. It turns out to be most convenient to work with the following L^2 -variant of the Kollár-Larsen deviation: $d_2(g)^2 = 2(\dim(V) - |\mathrm{Tr}(g)|)$ for $g \in GL(V)$ (see §2.2, in particular, Corollary 2.12 and Proposition 2.17(iii), for various notions of deviation and their relationships). Henceforth we say that a subgroup $G \leq GL(V)$ *has property \mathcal{P} up to scalars*, if there is a subgroup $H \leq GL(V)$ with property \mathcal{P} such that $Z(GL(V))G = Z(GL(V))H$.

The first main result of the paper is the following theorem which bounds the dimension of the representation in terms of the deviations of generators.

Theorem 1.1. *Let $G < GL(V)$ be a finite irreducible subgroup. Assume that there is a constant $C \geq 4$ such that, up to scalars, G is generated by some elements g_i with $d_2(g_i)^2 \leq C$, $1 \leq i \leq s$. Then one of the following holds.*

- (i) $\dim(V) \leq \mathfrak{f}(C) := \max\{4C^2/63, 40C\}$.
- (ii) $Z(G) \times A_n \leq G \leq (Z(G) \times A_n) \cdot 2$ and $\dim(V) = n - 1$, with A_n acting on V as on its deleted natural permutation module.
- (iii) G preserves a decomposition $V = V_1 \oplus \dots \oplus V_m$, with $\dim(V_i) \leq C/4$ and G inducing either S_m or A_m while permuting the m subspaces V_1, \dots, V_m .

One certainly expects the upper bound $\dim(V) \leq \mathfrak{f}(C)$ in Theorem 1.1(i) to have rather a theoretical than practical value. However, we notice that for C big enough (say $C \geq 630$), this bound is already quite close to be optimal, cf. Example 4.2. In general, as pointed out to the authors by Kollár, Theorem 1.1 should have interesting implications for differential geometry on symmetric spaces. Consider for instance locally symmetric spaces that behave locally like $GU_n(\mathbb{C})$: they are of the form $GU_n(\mathbb{C})/G$ for a finite subgroup $G < GU_n(\mathbb{C})$. Then the shortest closed geodesics in $GU_n(\mathbb{C})/G$ have length $2\pi \cdot \min_{1 \neq g \in G} \|g\|$, where $\|g\|$ is as defined in Definition 2.8. Here is one consequence of Theorem 1.1 in this context.

Corollary 1.2. *Let $G < GU(V)$ be a finite irreducible, primitive, tensor indecomposable subgroup. Assume that the shortest closed geodesics in $GU(V)/G$ have length $\leq L$. Then either one of the conclusions (i), (ii) of Theorem 1.1 holds for G with $C := \max\{4, L^2\}$, or $\dim(V) \leq (L \cdot |Z(G)|/2\pi)^2$.*

The next result shows that noncentral elements g of finite irreducible subgroups of $GL(V)$ usually have deviation $d_2(g)^2 \geq 4$, which implies that the condition $C \geq 4$ in Theorem 1.1 is natural.

Theorem 1.3. *Let $G < GL(V)$ be a finite primitive, irreducible subgroup. Let $d := \dim(V) \geq 2$, $g \in G \setminus Z(G)$, and set $\Delta(g) := \dim(V) - |\mathrm{Tr}(g)|$. If G is tensor induced, assume furthermore that g acts nontrivially on the set of tensor factors of V . Then one of the following statements hold.*

- (i) $d = 2$ and $\Delta(g) \geq (3 - \sqrt{5})/2$.
- (ii) $d = 3$ and $\Delta(g) \geq 3 - \sqrt{3}$.
- (iii) $d = 4$ and $\Delta(g) \geq 4 - 2\sqrt{2}$.
- (iv) $d \geq 5$ and either $\Delta(g) \geq 8 - 4\sqrt{2}$, or $\Delta(g) = 2$ and g is a scalar multiple of a reflection.
- (v) $V = A \otimes B$ is tensor decomposable as a G -module, $\dim(A) = 2$, $2 \leq \dim(B) \leq 6$, $g|_B$ is scalar, and $\Delta(g) \geq \dim(B) \cdot (3 - \sqrt{5})/2$.

The notion of *age* of elements of complex linear groups, see Definition 2.1, originates from the work of M. Reid [R1], [R2], [IR]. Its importance in algebraic geometry comes from the *Reid-Tai criterion* [R1]: *If the subgroup $G < GL_d(\mathbb{C})$ contains no complex reflections, then \mathbb{C}^d/G is terminal, resp. canonical, if and only if $\mathrm{age}(g) > 1$, resp. $\mathrm{age}(g) \geq 1$ for every $1 \neq g \in G$ (see e.g. [CK] for the definition of terminal and canonical singularities).* This implies in particular the following result of [IR]: *If G is a finite subgroup of $GL_d(\mathbb{C})$ and $f : X \rightarrow \mathbb{C}^d/G$ is a crepant resolution, then G contains elements g with $\mathrm{age}(g) \leq 1$.* Recall that, a resolution $f : X \rightarrow Y$ is said to be *crepant*, if $f^*K_Y = K_X$. Furthermore, in the profound programme of S. Mori to classify 3-dimensional algebraic varieties and in mirror symmetry, the singularities of type \mathbb{C}^d/G for some finite subgroup $G < GL_d(\mathbb{C})$ form a very good test class where many features of the general case can be tested in a computable setting. Recently, there has been a tremendous amount of research devoted to crepant resolutions. For instance, minimal models in Mori's programme utilize crepant maps. Crepant resolutions of quotients X/G of Calabi-Yau varieties X are also used in works on mirror symmetry (particularly as a way of obtaining *mirrors*). Physicists have long believed that string theories on a quotient space and on its crepant resolutions should be equivalent. Recent conjectures of Y. Ruan [Ru], and J. Bryan and T. Graber [BG] state that if $f : X \rightarrow Y$ is a crepant resolution, then quantum cohomology of X and of Y are essentially the same. More recently, Kollár and Larsen [KL] studied quotients X/G of a smooth projective Calabi-Yau variety X by a finite group G and showed in particular that the Kodaira dimension of X/G is controlled by whether $\mathrm{Stab}_x(G)$ contains non-trivial elements of age < 1 while acting on the tangent space $T_x X$ for some $x \in X$.

The next two theorems of the paper classify finite irreducible subgroups of $GL(V)$ that are generated by *junior* elements, that is, elements g with $0 < \mathrm{age}(g) \leq 1$, when $\dim(V) > 8$.

Theorem 1.4. *Let $V = \mathbb{C}^d$ with $d \geq 11$ and let $G < GL(V)$ be a finite irreducible subgroup. Assume that, up to scalars, G is generated by its elements with $\mathrm{age} \leq 1$.*

Then G contains a complex bireflection of order 2 or 3, and one of the following statements holds.

- (i) $Z(G) \times A_{d+1} \leq G \leq (Z(G) \times A_{d+1}) \cdot 2$, with A_{d+1} acting on V as on its deleted natural permutation module.
- (ii) G preserves a decomposition $V = V_1 \oplus \dots \oplus V_d$, with $\dim(V_i) = 1$ and G inducing either S_d or A_d while permuting the d subspaces V_1, \dots, V_d .
- (iii) $2|d$, and $G = D : S_{d/2} < GL_2(\mathbb{C}) \wr S_{d/2}$, a split extension of $D < GL_2(\mathbb{C})^{d/2}$ by $S_{d/2}$. Furthermore, if $g \in G \setminus D$ has $\text{age}(g) \leq 1$, then g is a bireflection (and $\text{age}(g) = 1$).

Theorem 1.5. *Let $V = \mathbb{C}^d$ with $d \geq 9$ and let $G < GL(V)$ be a finite irreducible subgroup. Assume that, up to scalars, G is generated by its elements with $\text{age} \leq 1$, and that G contains a scalar multiple of a non-central element g with $\text{age}(g) < 1$. Then one of the following statements holds.*

- (i) *One of the conclusions (i), (ii) of Theorem 1.4 holds, and G contains a scalar multiple of a complex reflection.*
- (ii) *The conclusion (iii) of Theorem 1.4 holds, and, modulo scalars, G cannot be generated by its elements of $\text{age} < 1$.*

The bound $d \geq 9$ in Theorem 1.5 is best possible, cf. Remark 5.15. In the case $4 \leq \dim(V) \leq 10$ of Theorem 1.4, the structure of the arising subgroups G is described in Proposition 5.16. On the contrary, from the group-theoretic viewpoint there is not much to say about the dimensions ≤ 3 : if $1 \neq g \in SL_d(\mathbb{C})$ has finite order, then $\text{age}(g) = 1$ if $d = 2$, and either $\text{age}(g)$ or $\text{age}(g^{-1}) = 1$ if $d = 3$.

A key ingredient in the proofs of Theorems 1.4 and 1.5 comes from Proposition 2.17 and its consequence Corollary 2.18, which relate the $\text{age}(g)$ to the L^2 -deviation $d_2(g)^2$ and thus allow us to invoke available results on character ratios for finite quasi-simple groups [G], [GM]. Also, see Theorem 5.9 for a lower bound on the age of any non-central element in finite linear groups. One should compare the latter result with the classical theorem of Blichfeldt stating that the shortest arc of S^1 which contains all eigenvalues of a non-central element in a finite primitive complex linear group has length at least $\pi/3$.

In the case the finite subgroup $G < GL(V)$ fixes a non-degenerate symplectic form on V , D. Kaledin [Ka] and M. Verbitsky [V] have shown that V/G can have a crepant resolution only when G is generated by complex bireflections. In general, however, it is not true that (non-central) elements of $\text{age} \leq 1$ are always complex bireflections (nor elements with fixed point subspace of codimension 2). In this regard, one of the main assertions of Theorem 1.4 is the existence of complex bireflections in the groups G satisfying the hypotheses of the theorem. If one knows that G is *generated* by complex bireflections (or G contains complex bireflections and is *quasiprimitive*), one can then appeal to available results on such groups, particularly [HW], [Hu], [Wa] (see also [Co]).

Interestingly, it was shown by V. Kac and K. Watanabe [KW], and independently by N. Gordeev [Go1], that if the ring $\text{Sym}(V)^G$ of G -invariants is a complete intersection for a finite group $G < GL(V)$, then G is generated by elements with fixed point subspace of codimension 2. The finite groups $G < GL(V)$ with $\text{Sym}(V)^G$ being a complete intersection have been classified by [Go2] and [N].

In a certain sense, Theorem 1.4 gives indications that crepant resolutions seem to occur mostly in low dimensions. Indeed, let $f : X \rightarrow \mathbb{C}^d/G$ be a crepant resolution, and let K be the normal subgroup of G generated by all elements of $\text{age} \leq 1$. Then by Theorem 1.4, for any irreducible summand V of the K -module \mathbb{C}^d , either $\dim(V) \leq 10$, or the action of K on V contains complex bireflections (of order 2 or 3), and so the quotient V/K should behave reasonably well from the point of view of algebraic geometry. (See [Ha] for the case of S_n acting on the sum $\mathbb{C}^n \oplus \mathbb{C}^n$ of two copies of the natural permutation module.) We formulate one consequence of our results in this regard:

Corollary 1.6. *Let $d \geq 11$ and let $G < GL_d(\mathbb{C})$ be a finite irreducible, primitive, tensor indecomposable subgroup. Assume that \mathbb{C}^d/G is not terminal (for instance, it has a crepant resolution). Then one of the following statements holds.*

- (i) $Z(G) \times A_{d+1} \leq G \leq (Z(G) \times A_{d+1}) \cdot 2$, with A_{d+1} acting on \mathbb{C}^d as on its deleted natural permutation module.
- (ii) All junior elements of G are central, and $|Z(G)| \geq d$.

Recall that (G, V) is a *basic non-RT pair* if $G < GL(V)$ is a finite irreducible subgroup and $G = \langle g^G \rangle$ for every non-central element $g \in G$ with $\text{age}(g) < 1$. This notion was first introduced in [KL] and is of importance for the geometry of quotients of Calabi-Yau varieties. Our third main result is concerned with this notion and is in fact predicted by results of [KL].

Theorem 1.7. *Let $G < GL(V)$ be a finite irreducible subgroup. Assume that, G contains non-central elements $g \in G$ with $\text{age}(g) < 1$, and that $G = \langle g^G \rangle$ for any such an element. Assume in addition that $\dim(V) > 4$. Then, up to scalars, G is a complex reflection group.*

Theorem 1.7 is not valid if $\dim(V) = 4$. Examples of 4-dimensional basic non-RT pairs which are not projectively equivalent to a c.r.g. are given in [KL]; see also Examples 3.7 and 5.8. One should also compare Theorem 1.7 with the classical result that \mathbb{C}^d/G is smooth if and only if the finite subgroup $G < GL_d(\mathbb{C})$ is a complex reflection group (see e.g. [B, Theorem V.5.4]).

There should be similar results for representations in positive characteristic (where we consider the eigenvalues of semisimple elements), and similar algebro-geometric applications. There are results which indicate that if $G < GL(V)$ with V finite dimensional over an algebraically closed field, then $k[V]^G$ being a polynomial ring, resp. a complete intersection, implies that G is generated by elements trivial on

a subspace of codimension 1, resp. on a subspace of codimension at most 2, cf. for instance [KM], [KW], [S]. Such groups have been classified (see [GS] for the last statement and references – also in [GS] finite and algebraic groups generated by symplectic reflections in all characteristics were classified). The authors have recently obtained some results on the values of Brauer characters which should be relevant.

2. PRELIMINARIES

Let $V = \mathbb{C}^n$ be endowed with standard Hermitian form (\cdot, \cdot) ; write $\|v\| = \sqrt{(v, v)}$ for any $v \in V$. Also let $S^1 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ and let $\mathcal{B}(V)$ be the collection of all orthonormal bases of V .

2.1. Age.

Definition 2.1. [IR], [R2]. Let $g \in GL(V)$ be conjugate to $\text{diag}(e^{2\pi i r_1}, \dots, e^{2\pi i r_n})$, where $0 \leq r_j < 1$. Then $\text{age}(g) = \sum_{j=1}^n r_j$.

Classical examples of non-scalar elements with $\text{age} < 1$ are: *reflections*, resp. *complex reflections* (or *pseudoreflections*), *bireflections*, and *complex bireflections*. These cases correspond to $(r_1, \dots, r_n) = (1/2, 0, \dots, 0)$, $(0 < r_1 < 1, 0, \dots, 0)$, $(1/2, 1/2, 0, \dots, 0)$, and $(0 < r_1 < 1, 1 - r_1, 0, \dots, 0)$, respectively. (Note that all complex bireflections considered in this paper have determinant 1.)

To deal with scalar multiples of linear transformations, it is also convenient to define

$$\text{age}^*(g) = \inf_{\lambda \in S^1} \text{age}(\lambda g)$$

for any (diagonalizable) $g \in GU(V)$.

First we record the following observations, which we usually apply to linear transformations of finite order (as elements of $GL(V)$).

Lemma 2.2. *The following statements hold for any $g \in GU(V)$.*

(i) $\text{age}(g)$ and $\text{age}^*(g)$ are well-defined, and constant on the $GU(V)$ -conjugacy class of g .

(ii) There is some $\mu \in S^1$ (of finite order, if $|g|$ is finite) such that $\text{age}^*(g) = \text{age}(\mu g)$. In particular, g is scalar if and only if $\text{age}^*(g) = 0$.

(iii) If $U \subseteq V$ is a g -invariant subspace then

$$\text{age}(g|_U) \leq \text{age}(g) = \text{age}(g|_U) + \text{age}(g|_{V/U}).$$

(iv) If $h \in GU(W)$, then

$$\text{age}^*(\text{diag}(g, h)) \geq \text{age}^*(g) + \text{age}^*(h), \quad \text{age}^*(g \otimes h) \geq \dim(W) \cdot \text{age}^*(g).$$

(v) If $h \in GU(V)$ and $gh = hg$, then

$$\text{age}(gh) \leq \text{age}(g) + \text{age}(h), \quad \text{age}^*(gh) \leq \text{age}^*(g) + \text{age}^*(h).$$

Proof. (i) and (iii) are obvious.

(ii) Let $e^{2\pi i r_1}, \dots, e^{2\pi i r_m}$, where $0 \leq r_1 < r_2 < \dots < r_m < 1$, be the distinct eigenvalues of g . Consider the function $f(t) := \text{age}(e^{-2\pi i t} \cdot g)$ on the interval $(0, 1]$. Note that f is decreasing on each of the intervals $(0, r_1]$, $(r_1, r_2]$, \dots , $(r_{m-1}, r_m]$, $(r_m, 1]$. It follows that $\text{age}^*(g) = \inf_{t \in (0, 1]} f(t)$ is attained as the value of f at one of the points $t = r_1, r_2, \dots, r_m, 1$. Thus we can take μ^{-1} to be either 1 or one of the eigenvalues of g , and so it has finite order in S^1 if $|g|$ is finite. (Also notice that if $m \geq 2$, then $\text{age}^*(g) \geq \min\{r_2 - r_1, 1 - (r_2 - r_1)\}$.)

(iv) Without loss we may assume that $h = \text{diag}(s_1, \dots, s_m)$ with $s_j \in S^1$, and consider any $\lambda \in S^1$. Then by (iii) we have

$$\begin{aligned} \text{age}(\lambda \cdot \text{diag}(g, h)) &= \text{age}(\lambda g) + \text{age}(\lambda h) \geq \text{age}^*(g) + \text{age}^*(h), \\ \text{age}(\lambda g \otimes h) &= \text{age}(\text{diag}(\lambda s_1 g, \dots, \lambda s_m g)) = \sum_{j=1}^m \text{age}(\lambda s_j g) \geq m \cdot \text{age}^*(g). \end{aligned}$$

(v) Without loss we may assume that

$$g = \text{diag}(e^{2\pi i r_1}, \dots, e^{2\pi i r_m}), \quad h = \text{diag}(e^{2\pi i s_1}, \dots, e^{2\pi i s_m}),$$

with $0 \leq r_j, s_j < 1$. Then $\text{age}(gh) \leq \sum_{j=1}^m (r_j + s_j) = \text{age}(g) + \text{age}(h)$. Next, by (ii) there are $\alpha, \beta \in S^1$ such that $\text{age}^*(g) = \text{age}(\alpha g)$ and $\text{age}^*(h) = \text{age}(\beta h)$. Now

$$\text{age}^*(gh) \leq \text{age}(\alpha \beta gh) = \text{age}(\alpha g \cdot \beta h) \leq \text{age}(\alpha g) + \text{age}(\beta h) = \text{age}^*(g) + \text{age}^*(h).$$

□

In fact, by the Chen-Ruan inequality [CR], Lemma 2.2(v) also holds without the condition $gh = hg$. Even more, the following inequality holds, where V^X denotes the common fixed point subspace for any subset $X \leq GL(V)$.

Theorem 2.3. [CR] (i) If $x, y \in GU(V)$, then

$$\text{age}(x) + \text{age}(y) - \text{age}(xy) + \dim V^{x,y} - \dim V^{xy} \geq 0.$$

(ii) If $x, y, z \in GU(V)$ and $xyz = 1$, then

$$\text{age}(x) + \text{age}(y) + \text{age}(z) \geq \dim(V) - \dim V^{x,y,z}.$$

This theorem follows from the existence of a cohomology theory developed in [CR], see also [Hep]. We will give an elementary proof of this result.

First we set up some notation. For $V = \mathbb{C}^n$ and $g \in GU(V)$, write $[g] = (v_1, \dots, v_n)$ where $0 \leq v_1 \leq \dots \leq v_n < 1$ and the eigenvalues of g are $e^{2\pi i v_j}$, $1 \leq j \leq n$.

Lemma 2.4. Let $\dim(V) > 1$ and $x, y \in GU(V)$, where x is a complex reflection with $[x] = (r, 0, \dots, 0)$, $0 < r < 1$, $[y] = (a_1, \dots, a_n)$ and $[xy] = (b_1, \dots, b_n)$. Let $H = \langle x, y \rangle$. Then the following conditions are equivalent:

- (i) H acts irreducibly.
- (ii) x and y have no common eigenvector.

(iii) The collection $\{a_1, \dots, a_n, b_1, \dots, b_n\}$ consists of $2n$ distinct elements.

Proof. If H acts reducibly, then $xy = y$ on some nontrivial H -invariant space, whence $a_j = b_k$ for some j, k . Thus (iii) implies (i), and certainly (i) implies (ii).

Now assume (ii); in particular, neither y nor xy has an eigenvector on $V^x = u^\perp$ (for some $0 \neq u \in V$). Note that $xu = e^{2\pi ir}u$. If $a_i = a_j$ for $i < j$, then y has a two dimensional eigenspace which therefore intersects V^x nontrivially, a contradiction. Similarly, we see that $b_i \neq b_j$. Suppose now that both xy and y have a common eigenvalue β . In this case, again by (ii) we can find $v, w \in u^\perp$ such that $y(u + v) = \beta(u + v)$ and $xy(u + w) = \beta(u + w)$; in particular, $y(u + w) = e^{-2\pi ir}\beta u + \beta w$. Thus $y(v - w) = \beta(1 - e^{-2\pi ir})u + \beta(v - w)$. Note that $|\beta| = 1$ and $\|y(v - w)\| = \|v - w\|$ as $y \in GU(V)$. It follows that $e^{2\pi ir} = 1$, a contradiction. \square

The key to Theorem 2.3 is the following beautiful result [BH, Cor. 4.7] on eigenvalue interlacing, see also [MOW].

Lemma 2.5. [BH] *Let $x, y \in GU(V)$, where x is a complex reflection with $[x] = (r, 0, \dots, 0)$, $0 < r < 1$, $[y] = (a_1, \dots, a_n)$ and $[xy] = (b_1, \dots, b_n)$. Assume that $a_j < a_{j+1}$ and $b_j < b_{j+1}$ for $1 \leq j < n$. Assume also that $a_j \neq b_k$ for any j, k . Then one of the following holds:*

- (a) $a_1 < b_1 < \dots < a_n < b_n$; or
- (b) $b_1 < a_1 < \dots < b_n < a_n$.

Note that in either case $|\text{age}(xy) - \text{age}(y)| < 1$ and so by considering determinants if (a) holds above, then $\text{age}(x) + \text{age}(y) = \text{age}(xy)$ while if (b) holds, then $\text{age}(x) + \text{age}(y) = \text{age}(xy) + 1$. In any case, $\text{age}(x) + \text{age}(y) \geq \text{age}(xy)$.

We now sketch an elementary proof of Lemma 2.5.

Proof. 1) Clearly, we may assume $\dim(V) > 1$. Let $V^x = u^\perp$ for some $0 \neq u \in V$. By Lemma 2.4, y cannot have any eigenvector in u^\perp or $\langle u \rangle_{\mathbb{C}}$. For $t \in \mathbb{R} \setminus \mathbb{Z}$, let $x(t)$ be the complex reflection with u^\perp as its reflecting hyperplane and $x(t)u = e^{2\pi it}u$. Also set $x(t) = 1_V$ if $t \in \mathbb{Z}$.

Now let $w(t) = x(t)y$ for $t \in \mathbb{R}$. Note that by construction, for any $t, t' \in \mathbb{R}$ with $t - t' \notin \mathbb{Z}$, $w(t)$ and $w(t')$ cannot have any common eigenvector. (Otherwise $x(t - t')$ and y have a common eigenvector v . This v must be either in u^\perp or $\langle u \rangle_{\mathbb{C}}$, contrary to the aforementioned property of y .) It then follows by Lemma 2.4 that, when $0 \leq t < t' < 1$, all the n eigenvalues of $w(t)$ are distinct, and $w(t)$ and $w(t')$ have no common eigenvalue.

2) Define $a_{kn+i} = a_i + k$ for $k \in \mathbb{Z}$. Also, let $[w(r)] = (b_1(r), \dots, b_n(r))$ and consider any $1 \leq j \leq n$. By the conclusion of 1), $a_i < b_j(r) < a_{i+1}$ for some $i \in \mathbb{Z}$. Note that the spectrum of $w(t)$ depends continuously on $t \in \mathbb{R}$. Hence, for t in some small neighborhood of r , the j^{th} -entry $b_j(t)$ of $[w(t)]$ satisfies $a_i < b_j(t) < a_{i+1}$. Let

$$X := \{s \mid r \leq s < 1, a_i < b_j(t) < a_{i+1} \text{ for all } t \in [r, s]\}.$$

We claim that $X = [r, 1)$. Indeed, let $f := \sup X \leq 1$ and assume $f < 1$. Then there is a sequence $\{s_n\} \subseteq X$ such that $\lim_{n \rightarrow \infty} s_n = f$. The spectrum continuity implies that $a_i \leq b_j(f) \leq a_{i+1}$. Since $0 < f < 1$, we must have $a_i < b_j(f) < a_{i+1}$. It is now easy to check that there is some $\epsilon > 0$ such that $f + \epsilon \in X$, a contradiction. Thus $f = 1$, which in turn implies that $X = [r, 1)$. Similarly,

$$\{s \mid 0 \leq s < r, \forall t \in [s, r], a_i < b_j(t) < a_{i+1}\} = (0, r].$$

We have shown that

$$(1) \quad a_i < b_j(t) < a_{i+1} \text{ for all } t \in (0, 1).$$

3) Replacing (x, y) by (x^{-1}, xy) if necessary, we may assume that $a_1 < b_1$. Note that $b_1(0) = a_1$. Hence $b_1(t)$ is close to a_1 when $t \in (0, 1)$ is small enough and so (1) implies that $a_1 < b_1(t) < a_2$ for all $t \in (0, 1)$; in particular, $b_1 = b_1(r) < a_2$. Also, $a_1 \leq b_1(1) = \lim_{t \rightarrow 1} b_1(t) \leq a_2$. Since $b_1(1)$ is some a_k , we get $b_1(1) \in \{a_1, a_2\}$. Moreover, if $b_1(1) = a_1 = b_1(0)$, then the continuity of $b_1(t)$ on $[0, 1]$ implies that some $w(t), w(t')$ with $0 < t < t' < 1$ have a common eigenvalue, contrary to the conclusion of 1). So $b_1(1) = a_2$.

4) Next, $b_2(0) = a_2$. If $b_2 = b_2(r) > a_2$, then, as above, (1) implies that $a_2 < b_2(t) < a_3$ for all $t \in (0, 1)$. Assume the contrary: $b_2 < a_2$. Again by (1) we must now have $a_1 < b_2(t) < a_2$ for all $t \in (0, 1)$. Arguing as in 3) we get $b_2(1) \in \{a_1, a_2\}$ and $b_2(1) \neq b_2(0) = a_2$, i.e. $b_2(1) = a_1$. On the other hand, $b_2(t) \geq b_1(t)$ for all $t \in [0, 1]$, whence $b_2(1) \geq b_1(1) = a_2$, a contradiction. We have shown that $a_2 < b_2(t) < a_3$ for all $t \in (0, 1)$. Continuing in the same fashion, we get $a_j < b_j(t) < a_{j+1}$ for all j and $t \in (0, 1)$. \square

Proof of Theorem 2.3. Notice that $\text{age}(z) = \dim(V) - \text{age}(xy) - \dim V^{xy}$, so (i) and (ii) are equivalent. Next, diagonalize x and then write x as a product of $m \leq n$ commuting complex reflections. To prove statement (i), we proceed by induction on m . First assume that x is a complex reflection. Let $H = \langle x, y \rangle$. If H does not act irreducibly, the result follows by induction on $\dim V$ (by writing $V = W \perp W^\perp$ where W is H -invariant). So assume this is not the case; in particular, $V^{x,y} = 0$. We need to prove that: $\text{age}(x) + \text{age}(y) \geq \text{age}(xy) + \dim V^{xy}$. By Lemma 2.4, no eigenspace of y or xy has dimension more than 1, and xy and y have no common eigenvalues. By Lemma 2.5 and the remarks before its proof, $\text{age}(x) + \text{age}(y) \geq \text{age}(xy)$. If xy has no trivial eigenvalue we are done. So we may assume that xy does have exactly one trivial eigenvalue, whence y has no trivial eigenvalue. Thus the case (b) of Lemma 2.5 holds and so $\text{age}(x) + \text{age}(y) = \text{age}(xy) + 1$ as desired.

For the induction step, write $x = sx'$ where s is a complex reflection and x' is a product of $m - 1$ complex reflections than x and $\text{age}(x) = \text{age}(s) + \text{age}(x')$. By the complex reflection case,

$$\text{age}(s) + \text{age}(x'y) + \dim V^{s,x'y} \geq \text{age}(xy) + \dim V^{xy}.$$

By induction,

$$\text{age}(x') + \text{age}(y) + \dim V^{x',y} \geq \text{age}(x'y) + \dim V^{x'y}.$$

Note that $V^{s,x'y} \cap V^{x',y} = V^{s,x',y} \subseteq V^{x,y}$ and $V^{s,x'y}, V^{x',y} \subseteq V^{x'y}$, whence

$$\dim V^{x'y} + \dim V^{x,y} \geq \dim V^{s,x'y} + \dim V^{x',y}.$$

The last three relations on dimensions readily imply

$$\text{age}(x) + \text{age}(y) = \text{age}(s) + \text{age}(x') + \text{age}(y) \geq \text{age}(xy) + \dim V^{xy} - \dim V^{x,y}. \square$$

2.2. The set-up (\star) . We are interested in finite subgroups of $GL(V)$ that contain nontrivial elements of $\text{age} < 1$, resp. ≤ 1 . Of course it would be very difficult to classify these groups without extra assumptions on them.

Lemma 2.6. *Let W be a finite dimensional vector space over \mathbb{C} and let $G < GL(W)$ be a finite subgroup containing a nontrivial element g with $\text{age}(g) < 1$, resp. $\text{age}(g) \leq 1$. Then there is a normal subgroup $K \triangleleft G$ and a nonzero K -invariant subspace V of W such that all the following conditions hold:*

- (i) K acts irreducibly on V ;
- (ii) $g \in K$ and $0 < \text{age}(g|_V) < 1$, resp. $0 < \text{age}(g|_V) \leq 1$;
- (iii) K is generated by the set of its elements whose restrictions to V have $\text{age} < 1$, resp. ≤ 1 .

In fact, if $0 \neq U \subseteq W$ is any K -submodule, then K is generated by the set of its elements whose restrictions to U have $\text{age} < 1$, resp. ≤ 1 .

Proof. Let \mathcal{X} denote the set of all nontrivial elements of G that have $\text{age} < 1$, resp. ≤ 1 , and define $K = \langle \mathcal{X} \rangle$. Then $K \triangleleft G$ and $K \ni g$. Decompose W into a direct sum $\oplus_{i=1}^s V_i$ of irreducible K -submodules. We may assume by Lemma 2.2 that $0 < \text{age}(g|_{V_1}) < 1$, resp. $0 < \text{age}(g|_{V_1}) \leq 1$. Let $\mathcal{Y} := \{h \in K \mid \text{age}(h|_{V_1}) < 1, \text{ resp. } \leq 1\}$. Observe that $\mathcal{X} \subseteq \mathcal{Y}$, whence $K = \langle \mathcal{Y} \rangle$. Thus $V := V_1$ satisfies (i) – (iii).

Next, let $0 \neq U \subseteq W$ be any K -submodule and let $\mathcal{Y}' := \{h \in K \mid \text{age}(h|_U) < 1, \text{ resp. } \leq 1\}$. Then again $\mathcal{X} \subseteq \mathcal{Y}'$ and so $K = \langle \mathcal{Y}' \rangle$. \square

Lemma 2.6 shows that it is natural to restrict our attention to the following set-up, which is slightly more general than the one considered in [KL]:

- (\star) : G is a finite irreducible subgroup of $\mathcal{G} = GL(V)$ and $Z(\mathcal{G})G = \langle \mathcal{X} \rangle$,
where $\mathcal{X} := \{g \in Z(\mathcal{G})G \mid 0 < \text{age}(g) < 1, \text{ resp. } 0 < \text{age}(g) \leq 1\}$.

The condition (\star) means that, up to scalars, the finite irreducible subgroup $G < GL(V)$ is generated by some nontrivial elements with $\text{age} < 1$, resp. ≤ 1 . In fact we can even assume that these generators are non-scalar:

Remark 2.7. *Assume G satisfies (\star) and $\dim(V) > 1$. Then $\mathcal{X}^* \neq \emptyset$ and $Z(\mathcal{G})G = Z(\mathcal{G})\langle \mathcal{X}^* \rangle$, where $\mathcal{X}^* := \{g \in G \mid 0 < \text{age}^*(g) < 1, \text{ resp. } 0 < \text{age}^*(g) \leq 1\}$. Indeed, if*

$\mathcal{X}^* = \emptyset$ then all $h \in \mathcal{X}$ are scalar and so is G , contradicting the condition $\dim(V) > 1$. Next, any $x \in G$ can be written as $g_1 \dots g_m$ with $g_i = \alpha_i h_i \in \mathcal{X}$, $\alpha_i \in S^1$, $h_i \in G$, and $h_i \in \mathcal{X}^*$ precisely when $i \in J$ for some subset $J \subseteq \{1, 2, \dots, m\}$. Then $x = \lambda \prod_{i \in J} h_i \in Z(\mathcal{G})\langle \mathcal{X}^* \rangle$ for $\lambda = \prod_{i=1}^m \alpha_i \cdot \prod_{j \notin J} h_j$.

2.3. Deviations. A natural invariant metric on $GU(V)$ is defined as follows:

Definition 2.8. Let $T \in GL(V)$ be conjugate to $\text{diag}(e^{2\pi i r_1}, \dots, e^{2\pi i r_n})$, where $0 \leq r_j < 1$. Then $\|T\| = (\sum_{j=1}^n \min\{r_j, 1 - r_j\}^2)^{1/2}$.

For our purposes it is more convenient to work with the following:

Definition 2.9. Let j be any positive number and let $T \in GU(V)$ be any unitary linear operator. Then

$$d_j(T) = \inf_{\lambda \in S^1, B \in \mathcal{B}(V)} \left(\sum_{b \in B} \|T(b) - \lambda b\|^j \right)^{1/j}.$$

This definition is a slight generalization of [KL, Definition 27] (where one takes $\lambda = 1$ instead of the infimum over all $\lambda \in S^1$). First we list some basic properties of $d_j(T)$.

Lemma 2.10. Let $A, T \in GU(V)$ and $\alpha \in S^1$. Then the following hold:

- (i) $d_j(T) = d_j(\alpha T)$;
- (ii) $d_j(T) = d_j(ATA^{-1})$;
- (iii) $d_j(T) = d_j(T^{-1})$.

Proof. (i) Clearly $\alpha T \in GU(V)$. Consider any $\lambda \in S^1$ and $B \in \mathcal{B}(V)$. Then

$$\sum_{b \in B} \|\alpha T(b) - \lambda b\|^j = \sum_{b \in B} \|T(b) - \alpha^{-1} \lambda b\|^j \geq d_j(T)^j.$$

Taking infimum over all $\lambda \in S^1$ and $B \in \mathcal{B}(V)$ we get $d_j(\alpha T) \geq d_j(T)$. Applying this inequality to $S := \alpha T$ and α^{-1} we obtain $d_j(T) = d_j(\alpha^{-1} S) \geq d_j(S) = d_j(\alpha T)$, and the claim follows.

(ii) Consider any $\lambda \in S^1$ and $B \in \mathcal{B}(V)$. Then $A^{-1}(B) \in \mathcal{B}(V)$, and

$$\sum_{b \in B} \|ATA^{-1}(b) - \lambda b\|^j = \sum_{c = A^{-1}b \in A^{-1}(B)} \|A(T(c) - \lambda c)\|^j = \sum_{c \in A^{-1}(B)} \|T(c) - \lambda c\|^j \geq d_j(T)^j.$$

Taking infimum over all $\lambda \in S^1$ and $B \in \mathcal{B}(V)$ we get $d_j(ATA^{-1}) \geq d_j(T)$. Applying this inequality to $S := ATA^{-1}$ and A^{-1} we obtain $d_j(T) = d_j(A^{-1}S(A^{-1})^{-1}) \geq d_j(S) = d_j(ATA^{-1})$, and the claim follows.

(iii) Consider any $\lambda \in S^1$ and $B \in \mathcal{B}(V)$. Then

$$\sum_{b \in B} \|T^{-1}(b) - \lambda b\|^j = \sum_{b \in B} \|\lambda^{-1} T(T^{-1}(b) - \lambda b)\|^j = \sum_{b \in B} \|T(b) - \lambda^{-1} b\|^j \geq d_j(T)^j.$$

Taking infimum over all $\lambda \in S^1$ and $B \in \mathcal{B}(V)$ we get $d_j(T^{-1}) \geq d_j(T)$. Applying this inequality to $S := T^{-1}$ we get $d_j(T) = d_j(S^{-1}) \geq d_j(S) = d_j(T^{-1})$, and so the claim follows. \square

Most of the time we will work with $d_j(T)$ where $j = 1$ or 2 .

Lemma 2.11. *For $T \in GU(V)$ the following hold:*

- (i) $d_2(T) \leq d_1(T) \leq \sqrt{\dim(V)} \cdot d_2(T)$;
- (ii) $d_1(T) \geq \dim(V) - |\text{Tr}(T)|$;
- (iii) $d_2(T)^2 = 2(\dim(V) - |\text{Tr}(T)|)$. Moreover, for any $B \in \mathcal{B}(V)$ we have

$$d_2(T) = \inf_{\lambda \in S^1} \left(\sum_{b \in B} \|T(b) - \lambda b\|^2 \right)^{1/2}.$$

Proof. (i) For any $\lambda \in S^1$ and $B \in \mathcal{B}(V)$ we have

$$d_2(T) \leq \left(\sum_{b \in B} \|T(b) - \lambda b\|^2 \right)^{1/2} \leq \sum_{b \in B} \|T(b) - \lambda b\|.$$

Taking infimum over all λ, B we get $d_2(T) \leq d_1(T)$. Next, again for any $\lambda \in S^1$ and $B \in \mathcal{B}(V)$ by the Cauchy-Schwarz inequality we have

$$d_1(T) \leq \sum_{b \in B} \|T(b) - \lambda b\| \leq \sqrt{\dim(V)} \cdot \left(\sum_{b \in B} \|T(b) - \lambda b\|^2 \right)^{1/2}.$$

Taking infimum over all λ, B we get $d_1(T) \leq \sqrt{\dim(V)} \cdot d_2(T)$.

(ii) Consider any $\lambda \in S^1$ and $B \in \mathcal{B}(V)$. Let $(a_{ij})_{1 \leq i, j \leq n}$ be the matrix of T in the basis B . Observe that

$$\begin{aligned} \sum_{b \in B} \|T(b) - \lambda b\| &= \sum_{l=1}^n \left(\sum_{k=1}^n |a_{kl} - \lambda \delta_{k,l}|^2 \right)^{1/2} \geq \sum_{l=1}^n |\lambda^{-1} a_{ll} - 1| \geq \sum_{l=1}^n \Re(1 - \lambda^{-1} a_{ll}) \\ &= n - \Re(\lambda^{-1} \sum_{l=1}^n a_{ll}) \geq n - |\lambda^{-1} \sum_{l=1}^n a_{ll}| = \dim(V) - |\text{Tr}(T)|. \end{aligned}$$

Taking infimum over all λ, B we arrive at the claim.

(iii) Consider an arbitrary $B \in \mathcal{B}(V)$ and let $A := (a_{ij})_{1 \leq i, j \leq n}$ be the matrix of T in the basis B . For any $\lambda \in S^1$ we have

$$\sum_{b \in B} \|T(b) - \lambda b\|^2 = \sum_{l=1}^n \sum_{k=1}^n |\delta_{k,l} - \lambda^{-1} a_{kl}|^2 = \sum_{1 \leq k, l \leq n} X_{kl} \bar{X}_{kl} = \text{Tr}({}^t \bar{X} \cdot X),$$

where $X := (\delta_{k,l} - \lambda^{-1} a_{kl})_{1 \leq k, l \leq n} = I_n - \lambda^{-1} A$. Since $T \in GU(V)$, there is a matrix C with ${}^t \bar{C} \cdot C = I_n$ and a diagonal matrix $E = \text{diag}(\epsilon_1, \dots, \epsilon_n)$ with $|\epsilon_k| = 1$ such that

$A = {}^t\bar{C}EC$. Then $X = {}^t\bar{C}DC$ for $D := I_n - \lambda^{-1}E = \text{diag}(1 - \alpha_1, \dots, 1 - \alpha_n)$ with $\alpha_i := \lambda^{-1}\epsilon_i$ (all of modulus 1). It follows that

$$(2) \quad \begin{aligned} \sum_{b \in B} \|T(b) - \lambda b\|^2 &= \text{Tr}({}^t\bar{X} \cdot X) = \text{Tr}({}^t\bar{C}{}^t\bar{D}C \cdot {}^t\bar{C}DC) = \text{Tr}({}^t\bar{D}D) \\ &= \sum_{l=1}^n |1 - \alpha_l|^2 = \sum_{l=1}^n (1 + |\alpha_l|^2 - 2\Re(\alpha_l)) = 2n - 2\Re(\sum_{l=1}^n \alpha_l). \end{aligned}$$

In particular,

$$(3) \quad \sum_{b \in B} \|T(b) - \lambda b\|^2 \geq 2n - 2\left|\sum_{l=1}^n \alpha_l\right| = 2n - 2\left|\sum_{l=1}^n \epsilon_l\right| = 2(n - |\text{Tr}(T)|).$$

Taking infimum over all λ, B we obtain $d_2(T)^2 \geq 2(n - |\text{Tr}(T)|)$.

Now, in the above computation we choose $\lambda = \lambda_0 := e^{i\theta}$, where $\text{Tr}(T) = re^{i\theta}$ and $|\text{Tr}(T)| = r \geq 0$. Then

$$\sum_{l=1}^n \alpha_l = \lambda_0^{-1} \sum_{l=1}^n \epsilon_l = \lambda_0^{-1} \text{Tr}(T) = e^{-i\theta} r e^{i\theta} = r = |\text{Tr}(T)|.$$

Then (2) implies that

$$d_2(T)^2 \leq \sum_{b \in B} \|T(b) - \lambda_0 b\|^2 = 2n - 2\Re\left(\sum_{l=1}^n \alpha_l\right) = 2(n - |\text{Tr}(T)|) \leq d_2(T)^2$$

Together with (3), this last inequality chain yields that

$$d_2(T)^2 = 2(n - |\text{Tr}(T)|) = \sum_{b \in B} \|T(b) - \lambda_0 b\|^2 = \inf_{\lambda \in S^1} \sum_{b \in B} \|T(b) - \lambda b\|^2.$$

□

The relationship between $\|T\|$ and $d_2(T)$ can be described as follows:

Corollary 2.12. *For $T \in GU(V)$ one has $4 \cdot \inf_{\lambda \in S^1} \|\lambda T\| < d_2(T) \leq 2\pi \cdot \|T\|$.*

Proof. We may assume that $T = \text{diag}(e^{2\pi i a_1}, \dots, e^{2\pi i a_n})$ in some basis $B \in \mathcal{B}(V)$, where $-1/2 \leq a_j < 1/2$; in particular, $\|T\| = (\sum_{j=1}^n a_j^2)^{1/2}$. It is easy to check that the function $(1 - \cos(x))/x^2$ is decreasing on $(0, \pi]$, whence $2/\pi^2 \leq (1 - \cos(x))/x^2 \leq 1/2$ for $-\pi \leq x \leq \pi$. Taking $x = 2\pi a_j$, we get $4/\pi^2 < |e^{2\pi i a_j} - 1|^2/4\pi^2 a_j^2 \leq 1$, whence $4\|T\| \leq (\sum_{b \in B} \|T(b) - b\|^2)^{1/2} \leq 2\pi\|T\|$. Now the statement follows by applying this inequality to λT for all $\lambda \in S^1$ and using Lemma 2.11(iii). □

Lemma 2.13. *Let $T_1, \dots, T_k \in GU(V)$. Then the following hold:*

- (i) $d_2(T_1 T_2 \dots T_k)^2 \leq k \cdot \sum_{i=1}^k d_2(T_i)^2$;
- (ii) $d_2(T_1 T_2 T_1^{-1} T_2^{-1})^2 \leq 4 \cdot \min\{d_2(T_1)^2, d_2(T_2)^2\}$.

Proof. (i) For any $\lambda_i \in S^1$, $B \in \mathcal{B}(V)$, and $b \in B$ we have

$$\begin{aligned} T_1 T_2 \dots T_k(b) - \lambda_1 \lambda_2 \dots \lambda_k b &= T_1 \dots T_{k-1}(T_k b - \lambda_1 b) + \lambda_1 T_1 \dots T_{k-2}(T_{k-1} b - \lambda_2 b) + \\ &\quad + \lambda_1 \lambda_2 T_1 \dots T_{k-3}(T_{k-2} b - \lambda_3 b) + \dots + \lambda_1 \dots \lambda_{k-1}(T_1 b - \lambda_k b). \end{aligned}$$

By the Cauchy-Schwarz inequality, $\|\sum_{i=1}^k v_i\|^2 \leq k \sum_{i=1}^k \|v_i\|^2$ for any $v_1, \dots, v_k \in V$. Since $|\lambda_i| = 1$ and T_i is unitary for all i , it now follows that

$$\|T_1 T_2 \dots T_k(b) - \lambda_1 \dots \lambda_k b\|^2 \leq k \cdot \sum_{i=1}^k \|T_i b - \lambda_i b\|^2,$$

But $\lambda_1 \dots \lambda_k \in S^1$, hence $d_2(T_1 \dots T_k)^2 \leq k \cdot \sum_{i=1}^k \sum_{b \in B} \|T_i b - \lambda_i b\|^2$. Taking infimum over all $\lambda_i \in S^1$ and applying Lemma 2.11(iii), we obtain $d_2(T_1 \dots T_k)^2 \leq k \cdot \sum_{i=1}^k d_2(T_i)^2$.

(ii) By (i) applied to $S := (T_1 T_2 T_1^{-1}) \cdot T_2^{-1}$ and by Lemma 2.10,

$$d_2(S)^2 \leq 2(d_2(T_1 T_2 T_1^{-1})^2 + d_2(T_2^{-1})^2) = 4d_2(T_2)^2.$$

Breaking up $S = T_1 \cdot T_2 T_1^{-1} T_2^{-1}$ and arguing similarly, we get $d_2(S)^2 \leq 4d_2(T_1)^2$. \square

Lemma 2.11(iii) and Lemma 2.13 yield the following inequalities which we believe to be new and nontrivial.

Corollary 2.14. *Let χ be any complex character of any finite group G and let $g_1, \dots, g_k \in G$. Then*

- (i) $(k^2 - 1)\chi(1) - k \sum_{i=1}^k |\chi(g_i)| + |\chi(\prod_{i=1}^k g_i)| \geq 0$;
- (ii) $3\chi(1) - 4|\chi(g_i)| + |\chi([g_1, g_2])| \geq 0$ for $i = 1, 2$. \square

Now we can prove an upper bound that links the dimension, covering number, and deviation together.

Lemma 2.15. *Let $V = \mathbb{C}^n$ with $n > 1$, $G < GL(V)$ a finite irreducible subgroup, and let $g \in G$. Assume that any element of $G/Z(G)$ is a product, of length at most β , of conjugates of $\bar{g} = gZ(G)$. Then $\dim(V) \leq (\beta d_2(g))^2/2$.*

Proof. By Weyl's unitarian trick we can equip V with a G -invariant Hermitian form and assume $G < GU(V)$. Consider any element $h \in G \setminus Z(G)$. Then $h = g_1 g_2 \dots g_k z$ with $g_i \in g^G$, $z \in Z(G)$, and $k \leq \beta$. By Schur's Lemma, z is scalar, hence $d_2(h) = d_2(h')$ for $h' := g_1 \dots g_k$ by Lemma 2.10(i). Next, by Lemma 2.10(ii) and Lemma 2.13(i), $d_2(h')^2 \leq k \sum_{i=1}^k d_2(g_i)^2 = k^2 d_2(g)^2$. It follows that $d_2(h)^2 \leq (\beta d_2(g))^2$.

Now by Burnside's theorem on zeros we can choose h such that $\text{Tr}(h) = 0$. By Lemma 2.11(iii), $d_2(h)^2 = 2n$ and so $2n \leq (\beta d_2(g))^2$. \square

Recall that a finite group G is *almost quasi-simple*, if $S \triangleleft G/Z(G) \leq \text{Aut}(S)$ for some finite non-abelian simple group S . For any such an S and any $x \in S$, let $\alpha(x)$ be the minimal number of $\text{Aut}(S)$ -conjugates of x which generate the subgroup

$\langle S, x \rangle$. A sharp upper bound on $\alpha(x)$ for $1 \neq x \in \text{Aut}(S)$ has been obtained in [GS]. We will need the following result of [GT2] that uses $\alpha(x)$ to bound the dimension of eigenspaces:

Lemma 2.16. [GT2, Lemma 3.2] *Let G be a finite almost quasi-simple group acting faithfully and irreducibly on a finite dimensional vector space V over a field \mathbb{F} , and let $g \in G \setminus Z(G)$. Then the dimension of any eigenspace of g on V is at most $\dim(V) - \dim(V)/\alpha(gZ(G))$. \square*

Next we prove key inequalities which relate the age of any element $g \in GU(V)$ to its deviations.

Proposition 2.17. *Let $g \in GU(V)$ and let X be a non-empty subset of eigenvalues of g . Let $m \geq 1$ be such that any $\lambda \in X$ occurs as an eigenvalue of g on V with multiplicity at least m . Also assume that the shortest arc of S^1 that contains X has length $\geq \delta > 0$. Then*

- (i) $2\pi \cdot \text{age}(g) - d_1(g) \geq m(\delta - 2\sin(\delta/2))$; in particular, $d_1(g) \leq 2\pi \cdot \text{age}(g)$.
- (ii) $4\pi \cdot \text{age}(g) - d_2(g)^2 \geq 2m(\delta - 1 + \cos(\delta))$.
- (iii) $d_2(g)^2 \leq (2.9)\pi \cdot \text{age}(g)$. In fact, if $\delta \geq \pi - \sin^{-1}(0.725)$ then

$$(2.9)\pi \cdot \text{age}(g) - d_2(g)^2 \geq m \{(1.45)\delta - 2(1 - \cos(\delta))\}.$$

Proof. Let g be represented by $\text{diag}(\epsilon_1, \dots, \epsilon_n)$ in a basis $B_0 \in \mathcal{B}(V)$, where $\epsilon_j = e^{2\pi i r_j}$, $0 \leq r_j < 1$; in particular, $\text{age}(g) = \sum_{j=1}^n r_j$.

(i) Consider the function $f(x) := 2\pi x - |e^{2\pi i x} - 1| = 2\pi x - 2\sin(\pi x)$ on $[0, \infty)$. Since $f'(x) = 2\pi - 2\pi \cos(\pi x) \geq 0$, f is increasing; also, $f(x) \geq 0$. Taking $\lambda = 1$ and $B = B_0$ in the proof of Lemma 2.11(ii), we see that $d_1(g) \leq \sum_{j=1}^n |\epsilon_j - 1|$. Hence

$$2\pi \cdot \text{age}(g) - d_1(g) \geq \sum_{j=1}^n f(r_j) \geq \sum_{j : \epsilon_j \in X} f(r_j).$$

Suppose that $0 \leq r_j \leq \delta'/2\pi < \delta/2\pi$ for all $\epsilon_j \in X$. Then X is contained in the arc (from 1 to $e^{i\delta'}$) of length $\delta' < \delta$, contrary to the assumption. So without loss we may assume that $\epsilon_1 \in X$ and $r_1 \geq \delta/2\pi$. Since ϵ_1 occurs as an eigenvalue of g with multiplicity $\geq m$, we get

$$2\pi \cdot \text{age}(g) - d_1(g) \geq \sum_{j : \epsilon_j \in X} f(r_j) \geq m f(r_1) \geq m f(\delta/2\pi) = m(\delta - 2\sin(\delta/2)).$$

(ii) Consider the function $h(x) := 4\pi x - |e^{2\pi i x} - 1|^2 = 4\pi x - 2(1 - \cos(2\pi x))$ on $[0, \infty)$. Since $h'(x) = 4\pi(1 - \sin(2\pi x)) \geq 0$, h is increasing, whence $h(x) \geq h(0) = 0$. Taking $\lambda = 1$ and $B = B_0$ in the proof of Lemma 2.11(iii), we see that $d_2(g)^2 \leq$

$\sum_{j=1}^n |\epsilon_j - 1|^2$. Hence

$$4\pi \cdot \text{age}(g) - d_2(g)^2 \geq \sum_{j=1}^n h(r_j) \geq \sum_{j : \epsilon_j \in X} h(r_j).$$

As in (i), we may assume without loss that $\epsilon_1 \in X$ and $r_1 \geq \delta/2\pi$. Since ϵ_1 occurs as an eigenvalue of g with multiplicity $\geq m$, we get

$$4\pi \cdot \text{age}(g) - d_2(g)^2 \geq \sum_{j : \epsilon_j \in X} h(r_j) \geq mh(r_1) \geq mh(\delta/2\pi) = 2m(\delta - 1 + \cos(\delta)).$$

(iii) Consider the function $t(x) := (1.45)x - |e^{ix} - 1|^2 = (1.45)x - 2(1 - \cos(x))$ on $[0, 2\pi]$. Since $t'(x) = 1.45 - 2\sin(x)$, t is increasing on $[0, \theta] \cup [\pi - \theta, 2\pi]$ and decreasing on $[\theta, \pi - \theta]$, where $\theta := \sin^{-1}(0.725)$. Now $t(0) = 0$ and $t(\pi - \theta) > 0.0018$, and so $t(x) \geq 0$ on $[0, 2\pi]$. As above, $d_2(g)^2 \leq \sum_{j=1}^n |\epsilon_j - 1|^2$, hence

$$(2.9)\pi \cdot \text{age}(g) - d_2(g)^2 \geq \sum_{j=1}^n t(2\pi r_j) \geq 0.$$

Next suppose that $\delta \geq \pi - \theta$. As in (i), we may assume without loss that $\epsilon_1 \in X$ and $r_1 \geq \delta/2\pi$. Since ϵ_1 occurs as an eigenvalue of g with multiplicity $\geq m$, $t(x) \geq 0$ and t is increasing on $[\pi - \theta, 2\pi]$, we see that $(2.9)\pi \cdot \text{age}(g) - d_2(g)^2 \geq m \cdot t(2\pi r_1) \geq m \cdot t(\delta)$. \square

Proposition 2.17 yields the following immediate consequence:

Corollary 2.18. *Let $G < GL(V)$ be a finite subgroup. Assume $g \in G$ is such that $\text{age}^*(g) \leq 1$. Then $d_1(g) \leq 2\pi$ and $d_2(g)^2 \leq (2.9)\pi < 9.111$. Furthermore, $\dim(V) - |\text{Tr}(g)| \leq (1.45)\pi < 4.556$. In fact,*

$$\dim(V) - |\text{Tr}(g)| < \begin{cases} 4.278, & \text{if } \delta \geq \pi, \\ 3.632, & \text{if } \delta \geq 6\pi/5, \\ 3.019, & \text{if } \delta \geq 4\pi/3, \\ 2.676, & \text{if } \delta \geq 7\pi/5, \\ 2.139, & \text{if } \delta \geq 3\pi/2, \end{cases}$$

where δ is the length of the shortest arc of S^1 that contains all eigenvalues of g .

Proof. We apply Lemma 2.11(iii), and Proposition 2.17(i), (iii), with $X = \text{Spec}(\lambda g, V)$ for any $\lambda \in S^1$. Then the claims follow by taking infimum over all $\lambda \in S^1$. \square

Taking $g = \text{diag}(1, 1, \dots, 1, -1, -1) \in GL_n(\mathbb{C})$ with $n \geq 4$, we see that $\text{age}(g) = 1$ and $d_2(g)^2 = 8$. In fact, the complex reflection $g = \text{diag}(e^{2\pi i/3}, 1, \dots, 1) \in GL_n(\mathbb{C})$ has $\text{age}(g) = 1/3$ and $d_2(g)^2 = 2(n - \sqrt{n^2 - 3n + 3})$. Hence, when $n \rightarrow \infty$, $d_2(g)^2 \rightarrow 3$, yielding $d_2(g)^2/\text{age}(g) \rightarrow 9$. Thus the constant $(2.9)\pi \approx 9.111$ in Proposition 2.17(iii) and Corollary 2.18 is quite good.

Lemma 2.19. *Let $G < GL(V)$ be irreducible, primitive, and tensor indecomposable on V , with a normal subgroup $L \triangleleft G$ such that $L' \not\leq Z(G)$.*

- (i) *For any non-scalar $g \in G$, there exists $h \in L \setminus Z(L)$ such that $d_2(h)^2 \leq 4d_2(g)^2$.*
- (ii) *Assume furthermore that $\dim(V) > 1$ and that G satisfies the set-up (\star) . Then there exists $h \in L \setminus Z(L)$ such that $d_2(h)^2 < 36.444$.*

Proof. (i) By [GT3, Lemma 2.5], L' is irreducible on V . We claim that there exists $u \in L$ such that $[g, u] \notin Z(L)$. Assume the contrary: $[g, u] \in C_G(L)$ for any $u \in L$. Then for any $u, v \in L$ we have $[[u, v], g] = ([v, g], u) \cdot [[g, u], v]^{-1} = 1$, whence $[g, L'] = 1$. By Schur's Lemma, the irreducibility of L' on V now implies that g is scalar, a contradiction. Now we define $h = [g, u] \in L \setminus Z(L)$, and we are done by Lemma 2.13(ii).

(ii) If every $g \in \mathcal{X}$ acts scalarly on V , then so does G . But in this case $\dim(V) = 1$, a contradiction. Hence at least one $g \in \mathcal{X}$ is non-scalar, and has $\text{age} \leq 1$. Now the claim follows from (i) and Corollary 2.18. \square

2.4. Elements of small order. To estimate the age of elements of small order, we will need the following two statements.

Lemma 2.20. *Assume $g \in GU(V)$ is conjugate to*

$$\text{diag}(\alpha_1, -\alpha_1, \alpha_2, -\alpha_2, \dots, \alpha_m, -\alpha_m, \beta_1, \dots, \beta_s).$$

Then $\text{age}^(g) \geq m/2$. Moreover, if $\text{age}^*(g) = m/2$ then g has exactly two distinct eigenvalues.*

Proof. Suppose $\text{age}^*(g) \leq m/2$. Then $\text{age}(\mu g) \leq m/2$ for some $\mu \in S^1$ by Lemma 2.2(ii). Note that the contribution of the pair $(\mu\alpha_i, -\mu\alpha_i)$ to $\text{age}(g)$ is at least $1/2$, and it equals $1/2$ precisely when $\alpha_i = \pm\mu^{-1}$. Next, the contribution of $\mu\beta_j$ to $\text{age}(g)$ is at least 0 , and it equals 0 precisely when $\beta_j = \mu^{-1}$. Hence the statements follow. \square

Lemma 2.21. *Let $g \in GU(V)$ be a non-scalar element of $\text{age} \leq 1$, $\dim(V) \geq 4$, and let λg have order $1 < m \leq 5$ for some $\lambda \in S^1$. Then there is some $\mu \in S^1$ such that either μg is a complex reflection, or one of the following statements holds for a suitable choice of $i = \sqrt{-1}$.*

- (i) $m = 2$, and g is a bireflection.
- (ii) $m = 3$, and one of the following holds, where $\omega = e^{2\pi i/3}$.
 - (a) μg is conjugate to $\text{diag}(\omega, \omega, 1, \dots, 1)$.
 - (b) $\text{age}(g) = 1$, and g is conjugate to $\text{diag}(\omega, \omega^2, 1, \dots, 1)$ or $\text{diag}(\omega, \omega, \omega, 1, \dots, 1)$.
- (iii) $m = 4$, and one of the following holds.
 - (a) μg is conjugate to one of the elements

$$\text{diag}(i, i, 1, \dots, 1), \text{diag}(i, i, i, 1, \dots, 1), \text{diag}(i, -1, 1, \dots, 1)$$

- (b) $\text{age}(g) = 1$, and g is conjugate to one of the elements

$$\text{diag}(i, -i, 1, \dots, 1), \text{diag}(-1, -1, 1, \dots, 1), \text{diag}(i, i, -1, 1, \dots, 1), \text{diag}(i, i, i, i, 1, \dots, 1)$$

(iv) $m = 5$, and one of the following holds, where $\epsilon = e^{2\pi i/5}$.

(a) μg is conjugate to one of the elements

$$\begin{aligned} & \text{diag}(\epsilon, \epsilon, 1, \dots, 1), \text{diag}(\epsilon^2, \epsilon^2, 1, \dots, 1), \\ & \text{diag}(\epsilon, \epsilon, \epsilon, 1, \dots, 1), \text{diag}(\epsilon, \epsilon, \epsilon, \epsilon, 1, \dots, 1), \\ & \text{diag}(\epsilon, \epsilon^2, 1, \dots, 1), \text{diag}(\epsilon, \epsilon^3, 1, \dots, 1), \text{diag}(\epsilon, \epsilon, \epsilon^2, 1, \dots, 1). \end{aligned}$$

(b) $\text{age}(g) = 1$, and g is conjugate to one of the elements

$$\begin{aligned} & \text{diag}(\epsilon, \epsilon^4, 1, \dots, 1), \text{diag}(\epsilon^2, \epsilon^3, 1, \dots, 1), \text{diag}(\epsilon, \epsilon^2, \epsilon^2, 1, \dots, 1), \\ & \text{diag}(\epsilon, \epsilon, \epsilon^3, 1, \dots, 1), \text{diag}(\epsilon, \epsilon, \epsilon, \epsilon^2, 1, \dots, 1), \text{diag}(\epsilon, \epsilon, \epsilon, \epsilon, \epsilon, 1, \dots, 1). \end{aligned}$$

Proof. The proofs of all these statements are similar, and we display it for (iv). By the assumption, there is some $t \in [0, 1/5)$, and integers $a, b, c, d, e \geq 0$ such that $a + b + c + d + e = \dim(V) \geq 4$ and

$$1 \geq \text{age}(g) = at + b(t + 1/5) + c(t + 2/5) + d(t + 3/5) + e(t + 4/5);$$

in particular, $b + 2c + 3d + 4e \leq 5$. Now the statement (iv) follows by an exhaustive enumeration. \square

2.5. Character ratios. We will need the following result of Gluck and Magaard, cf. [G] and [GM].

Proposition 2.22. *Let G be a finite group, $\chi \in \text{Irr}(G)$ be of degree > 1 , and $g \in G \setminus Z(G)$.*

(i) [GM, Theorem 2.4] *Assume G is a finite quasi-simple group, not A_n nor $2A_n$ with $n \geq 10$. Then $|\chi(g)/\chi(1)| \leq 19/20$.*

(ii) [GM, Theorem 1.6] *Let $G = S_n$ or A_n with $n \geq 5$, and let $c(g)$ be the number of cycles of the permutation g . Then $|\chi(g)/\chi(1)| \leq 1/2 + c(g)/2n$.* \square

Next we address the character ratios for spin representations of $2A_n$ and $2S_n$.

Lemma 2.23. *Let $G = 2S_n$ or $2A_n$ with $n \geq 6$, $\chi \in \text{Irr}(G)$ a faithful character of G , and let $g \in G \setminus Z(G)$. Then $|\chi(g)/\chi(1)| \leq 7/8$.*

Proof. Since $g \notin Z(G) = C_G(G')$ and since $G' = 2A_n$ is generated by commutators $[x, y]$, with x, y being inverse images of all 3-cycles, there exists an inverse image t of a 3-cycle such that $h := [g, t] \notin Z(G)$. Observe that $h = gtg^{-1} \cdot t^{-1}$ projects onto the product of two 3-cycles. It follows that (a G -conjugate of) h is contained in a natural subgroup $K \cong 2A_6$ of G . (See [GM, Lemma 2.5] for a similar argument.) Clearly, $h \notin Z(K)$ since $[g, t] \notin Z(G)$. Also, the restriction $\chi|_K$ is a sum of faithful irreducible characters of K . Inspecting [Atlas], one can check that $|\chi(h)| \leq \chi(1)/2$, and so $d_2(h)^2 \geq \chi(1)$. It now follows from Lemma 2.13(ii) that $d_2(g)^2 \geq \chi(1)/4$, whence $|\chi(g)/\chi(1)| \leq 7/8$ by Lemma 2.11 (iii). \square

2.6. Tensor decomposable and tensor induced modules. First we recall a well-known remark:

Lemma 2.24. *Let G be a finite irreducible subgroup of $GL(W)$. Assume that there is a tensor decomposition $W = U \otimes V$ such that $G < GL(U) \otimes GL(V)$. Then there is a finite central extension $1 \rightarrow Z \rightarrow \tilde{G} \rightarrow G \rightarrow 1$ of G and irreducible representations $\Phi : \tilde{G} \rightarrow GL(U)$ and $\Psi : \tilde{G} \rightarrow GL(V)$ such that $g = \Phi(\tilde{g}) \otimes \Psi(\tilde{g})$ for any $g = \tilde{g}Z \in G$.*

Proof. First we observe that if $a \otimes b = c \otimes d$ for some $a, c \in GL(U)$ and $b, d \in GL(V)$, then there is some $\gamma \in \mathbb{C}^\times$ such that $a = \gamma c$ and $b = \gamma^{-1}d$. Now, by the hypothesis, there are maps $A : G \rightarrow GL(U)$ and $B : G \rightarrow GL(V)$ such that $g = A(g) \otimes B(g)$ for any $g \in G$. If $h \in G$, then

$$A(gh) \otimes B(gh) = gh = (A(g) \otimes B(g)) \cdot (A(h) \otimes B(h)) = (A(g) \cdot A(h)) \otimes (B(g) \cdot B(h)).$$

By our observation, we see that $A(gh) = \lambda(g, h)A(g)A(h)$ for some 2-cocycle $\lambda : G \times G \rightarrow \mathbb{C}^\times$, and so A is a projective (irreducible) representation of G . Thus A lifts to a linear representation $\Phi : \tilde{G} \rightarrow GL(U)$ of a finite central extension \tilde{G} of G : $A(g) = \alpha(\tilde{g})\Phi(\tilde{g})$, where $\alpha : \tilde{G} \rightarrow \mathbb{C}^\times$ and $g = \tilde{g}Z$. Now it is easy to check that the map $\Psi : \tilde{G} \rightarrow GL(V)$, defined by $\Psi(\tilde{g}) = \alpha(\tilde{g})B(g)$ for $g = \tilde{g}Z$, is a group homomorphism, and $g = \Phi(\tilde{g}) \otimes \Psi(\tilde{g})$. \square

Lemma 2.24 shows that if a finite irreducible subgroup G of $GL(V)$ preserves a tensor decomposition of V , then we may (and will) view V as the tensor product of two modules for some central extension \tilde{G} of G , and then replace G by \tilde{G} .

Let $V = \mathbb{C}^d$ be a G -module, which is tensor induced. This means that there is a tensor decomposition $V = V_1^{\otimes m}$ such that (the action of) G (on V) is contained in $GL(V_1)^{\otimes m} : \mathbf{S}_m$, with \mathbf{S}_m naturally permuting the m tensor factors of V . (Note that we do not claim that $G \leq H^{\otimes m} : \mathbf{S}_m$ for some finite subgroup $H \in GL(V_1)$.)

Lemma 2.25. *Under the above assumptions, assume G is finite and $g \in G$ projects onto $h \in \mathbf{S}_m$, a product of s disjoint cycles. Then $|\text{Tr}(g)| \leq \dim(V_1)^s$.*

Proof. First we observe that if $y = a \otimes b$ has finite order for $a \in GL(U)$ and $b \in GL(V)$, then there is some $\delta \in \mathbb{C}^\times$ such that both $c := \delta^{-1}a$ and $d := \delta b$ have finite order, and $y = c \otimes d$. (Indeed, $I = y^N = a^N \otimes b^N$, where we use I to denote any identity matrix. So by the first sentence of the proof of Lemma 2.24, $a^N = \gamma I$ and $b^N = \gamma^{-1}I$ for some $\gamma \in \mathbb{C}^\times$ and $0 < N \in \mathbb{Z}$. Now choose δ to be an N^{th} -root of γ .)

In the case $s > 1$, conjugating g with a suitable element in \mathbf{S}_m we may assume that g preserves a tensor decomposition of V . Using the above observation and proceeding by induction on s , we may assume that $s = 1$ and $h = (1, 2, \dots, m)$. Now $g = hb$ with $b = B_1 \otimes B_2 \otimes \dots \otimes B_m$ and $B_i \in GL(V_1)$. Then one can check (see also [GI]) that $\text{Tr}(g) = \text{Tr}(B_1 B_2 \dots B_m)$. Since G and \mathbf{S}_m are finite, there is some integer $N > 1$ such

that $g^N = h^N = \text{Id}$. Since $I = h^N = g^N = h^N b^{h^{N-1}} b^{h^{N-2}} \dots b^h b$ (where $b^x := x^{-1}bx$), we have

$$I = g^N = (B_2 B_3 \dots B_m B_1)^{N/m} \otimes (B_3 B_4 \dots B_m B_1 B_2)^{N/m} \otimes \dots \otimes (B_1 B_2 \dots B_m)^{N/m}.$$

Pick an arbitrary eigenvalue λ of $v := B_1 B_2 \dots B_m$. Note that all the matrices $B_2 B_3 \dots B_m B_1$, $B_3 B_4 \dots B_m B_1 B_2$, \dots are conjugate to v . Hence $\underbrace{\lambda^{N/m} \dots \lambda^{N/m}}_m =$

λ^N is an eigenvalue of $g^N = I$. We have shown that each eigenvalue of v is an N^{th} -root of unity and so it has absolute value 1. Hence $|\text{Tr}(g)| = |\text{Tr}(v)|$ is at most the size of v , which is $\dim(V_1)$. \square

We will also need the following technical statement:

Lemma 2.26. *Let \mathcal{C} be a collection of finite simple groups and let G be any finite group. Then G has a unique normal subgroup R such that*

- (i) *every composition factor of R belongs to \mathcal{C} ; and*
- (ii) *If $N \triangleleft G$ and every composition factor of N belongs to \mathcal{C} , then $N \leq R$.*

Furthermore, R is a characteristic subgroup of G .

Proof. Let \mathcal{X} be the collection of all normal subgroups $N \triangleleft G$ with the property that all composition factors of N belong to \mathcal{C} . For any $M, N \in \mathcal{X}$, $MN \triangleleft G$, and every composition factor of MN also belongs to \mathcal{C} since $MN/N \cong M/(M \cap N)$, whence $MN \in \mathcal{X}$. Now the subgroup $R = \prod_{N \in \mathcal{X}} N$ clearly satisfies (i) and (ii). Let $\varphi \in \text{Aut}(G)$. Then $\varphi(R) \triangleleft G$ and $\varphi(R) \in \mathcal{X}$ since $\varphi(R) \cong R$. By (ii), $\varphi(R) = R$. \square

3. PROOF OF THEOREM 1.3

3.1. Reduction to the almost quasi-simple case.

Proposition 3.1. *It suffices to prove Theorem 1.3 for the case G is an almost quasi-simple group which is irreducible, primitive, tensor indecomposable, and not tensor induced on V .*

Proof. Let χ denote the character of G afforded by V .

(i) First we consider the case G is tensor induced on V : $V = V_1 \otimes V_2 \otimes \dots \otimes V_m$, with $\dim(V_i) = a > 1$ and G permutes the m tensor factors V_1, \dots, V_m (transitively). By assumption, g acts nontrivially on the set $\{V_1, \dots, V_m\}$. Hence, $|\chi(g)| \leq a^{m-1} \leq d/2$ by Lemma 2.25. Now if $d = a^m \geq 8$ then $\Delta(g) \geq d/2 \geq 4$. On the other hand, if $d = a^m < 8$, then $d = 4$ and $\Delta(g) \geq d/2 = 2$.

(ii) Now assume that we are in the extraspecial case (i.e. the case (iii) of [GT3, Proposition 2.8]). Then $d = p^m$ for some prime p and some integer $m \geq 2$. By [GT1, Lemma 2.4], $|\chi(g)| \leq p^{m-1/2} \leq d/\sqrt{2}$. In particular, if $d \geq 8$, then $\Delta(g) \geq d(1 - 1/\sqrt{2}) \geq 8 - 4\sqrt{2}$. If $d = 5$ or 7 , then $\Delta(g) \geq d - \sqrt{d} \geq 5 - \sqrt{5} > 8 - 4\sqrt{2}$. If $d = 4$, then $\Delta(g) \geq 4(1 - 1/\sqrt{2}) = 4 - 2\sqrt{2}$. If $d = 3$, then $\Delta(g) \geq 3 - \sqrt{3}$, and if $d = 2$, then $\Delta(g) \geq 2 - \sqrt{2} > (3 - \sqrt{5})/2$.

(iii) Next we consider the tensor decomposable case: $V = V_1 \otimes \dots \otimes V_m$, where G is tensor indecomposable and primitive on V , $\dim(V_i) \geq 2$, and $m \geq 2$. By [GT3, Proposition 2.8] and by the hypothesis, we may assume that Theorem 1.3 holds for g acting on V_i as long as $g|_{V_i}$ is not scalar. Let α_i be the character afforded by V_i .

Suppose that there is some j such that $\dim(V_j) \geq 3$ and $g|_{V_j}$ is non-scalar. Then $\alpha_j(1) - |\alpha_j(g)| \geq 4 - \sqrt{8}$ by Theorem 1.3 applied to (G, g, V_j) . Hence

$$\chi(1) - |\chi(g)| \geq \chi(1) - \frac{\chi(1)}{\alpha_j(1)} |\alpha_j(g)| = \frac{\chi(1)}{\alpha_j(1)} (\alpha_j(1) - |\alpha_j(g)|) \geq 2(4 - \sqrt{8}) = 8 - 4\sqrt{2},$$

as required. So we may assume that $\dim(V_i) = 2$ whenever $g|_{V_i}$ is non-scalar. But g is non-scalar, so without loss we may suppose that $\dim(V_1) = 2$ and $g|_{V_1}$ is non-scalar. By Theorem 1.3 applied to (G, g, V_1) we have $\alpha_1(1) - |\alpha_1(g)| \geq (3 - \sqrt{5})/2$. Arguing as above, we obtain that

$$\Delta(g) = \chi(1) - |\chi(g)| \geq \frac{\chi(1)}{\alpha_1(1)} (\alpha_1(1) - |\alpha_1(g)|) \geq d(3 - \sqrt{5})/4.$$

If $d \geq 13$ in addition, then in fact $d \geq 14$ and $\Delta(g) \geq 7(3 - \sqrt{5})/2 > 8 - 4\sqrt{2}$. If $d = 6$ or 10 , then $m = 2$, $\dim(V_2) = 3$ or 5 , and so $g|_{V_2}$ is scalar, whence we arrive at the conclusion (v) of Theorem 1.3. The same holds if $m = 2$ and $d \in \{8, 12\}$. We also arrive at the same conclusion when $d = 4$, as otherwise $\Delta(g) \geq 4 - ((1 + \sqrt{5})/2)^2 > 4 - \sqrt{8}$. Finally, consider the case where $d \in \{8, 12\}$ but $m > 2$; that is, $m = 3$. Then we may assume that $\dim(V_2) = 2$ and $g|_{V_2}$ is not scalar (as otherwise Theorem 1.3(v) holds). As in the case $d = 4$, we get $\alpha_1(1)\alpha_2(1) - |\alpha_1(g)\alpha_2(g)| > 4 - \sqrt{8}$, whence

$$\Delta(g) = \chi(1) - |\chi(g)| \geq \frac{\chi(1)}{\alpha_1(1)\alpha_2(1)} (\alpha_1(1)\alpha_2(1) - |\alpha_1(g)\alpha_2(g)|) > 8 - 4\sqrt{2}.$$

We are done by [GT3, Proposition 2.8]. \square

Throughout the rest of this section we will assume that G is an almost quasi-simple group. In fact, we will prove more than we need for the proof of Theorem 1.3: we will describe all triples (G, V, g) , where

- $G < GL(V)$ is an almost quasi-simple, irreducible, primitive,
- (♣) : tensor indecomposable subgroup, $g \in G \setminus Z(G)$, and
- either $0 < \text{age}^*(g) \leq 1$, or $\Delta(g) := \dim(V) - |\text{Tr}(g)| \leq 8 - 4\sqrt{2}$.

As usual, we denote by χ the character of G afforded by V , $L := G^{(\infty)}$, $S := L/Z(L)$. The set-up (♣) implies that $\chi|_L$ is irreducible, and that

$$(4) \quad \Delta(g) = \chi(1) - |\chi(g)| < 4.556$$

by Corollary 2.18.

3.2. Alternating groups. First we dispose of the case $S = A_n$ with $n \geq 8$. For $1 \leq k \leq n-1$, let $R_n(k)$ denote the set of partitions $\lambda \vdash n$, where either λ or the conjugate partition λ^* has the form $(n-k, \mu)$ for some $\mu \vdash k$. We will need the following statement which follows from the main result of [Ra]:

Lemma 3.2. *Let $\rho = \rho^\lambda \in \text{Irr}(S_n)$ be labeled by the partition $\lambda \vdash n$.*

- (i) *If $n \geq 15$, then either $\rho(1) \geq n(n-1)(n-5)/6$, or $\lambda \in \cup_{k=1}^2 R_n(k)$.*
- (ii) *If $n \geq 22$, then either $\rho(1) \geq n(n-1)(n-2)(n-7)/24$, or $\lambda \in \cup_{k=1}^3 R_n(k)$. \square*

We will now estimate $\rho^\lambda(g)$. Let \mathbf{t} denote the transposition $(1, 2) \in S_n$.

Lemma 3.3. *Let $1 + \alpha(g)$ denote the number of fixed points of the permutation $g \in S_n$, and let $n \geq 9$. Then*

$$\rho^\lambda(g) = \begin{cases} (\alpha(g)^2 - \alpha(g^2))/2, & \lambda = (n-2, 1^2), \\ (\alpha(g)^2 + \alpha(g^2))/2 - \alpha(g) - 1, & \lambda = (n-2, 2), \\ (\alpha(g)^3 - 3\alpha(g)\alpha(g^2) + 2\alpha(g^3))/6, & \lambda = (n-3, 1^3), \\ (\alpha(g)^3 - \alpha(g^3))/3 - \alpha(g)^2 + 1, & \lambda = (n-3, 2, 1), \\ (\alpha(g)^3 + 3\alpha(g)\alpha(g^2) + 2\alpha(g^3))/6 - \alpha(g)^2 - \alpha(g), & \lambda = (n-3, 3). \end{cases}$$

In particular, if $g \neq 1$ then $|\rho^\lambda(g)| \leq \rho^\lambda(\mathbf{t})$ for any of the above λ .

Proof. It is well known that $\text{Sym}^2(\alpha) = \rho^{(n-2,2)} + \rho^{(n-1,1)} + \rho^{(n)}$, $\wedge^2(\alpha) = \rho^{(n-2,1^2)}$, and $\wedge^3(\alpha) = \rho^{(n-3,1^3)}$, cf. [FH] for instance. Using the Littlewood-Richardson rule, one can see that

$$\begin{aligned} & \left(\text{Ind}_{S_{n-1}}^{S_n}(\rho^{(n-1)}) \right) \otimes \wedge^2(\alpha) = \text{Ind}_{S_{n-1}}^{S_n} \left((\rho^{(n-2,1^2)})|_{S_{n-1}} \right) = \\ & = \text{Ind}_{S_{n-1}}^{S_n} \left(\rho^{(n-3,1^2)} + \rho^{(n-2,1)} \right) = \rho^{(n-3,1^3)} + \rho^{(n-3,2,1)} + 2\rho^{(n-2,1^2)} + \rho^{(n-2,2)} + \rho^{(n-1,1)} \end{aligned}$$

and so

$$\alpha \otimes \wedge^2(\alpha) = \rho^{(n-3,1^3)} + \rho^{(n-3,2,1)} + \rho^{(n-2,1^2)} + \rho^{(n-2,2)} + \rho^{(n-1,1)}.$$

Similarly,

$$\alpha \otimes \rho^{(n-2,2)} = \rho^{(n-3,3)} + \rho^{(n-3,2,1)} + \rho^{(n-2,1^2)} + \rho^{(n-2,2)} + \rho^{(n-1,1)}.$$

It now follows that

$$\begin{aligned} \rho^{(n-3,2,1)} &= \alpha \otimes \wedge^2(\alpha) - \wedge^3(\alpha) - \alpha \otimes \alpha + 1, \\ \rho^{(n-3,3)} &= \alpha \otimes \rho^{(n-2,2)} - \alpha \otimes \wedge^2(\alpha) + \wedge^3(\alpha), \end{aligned}$$

and we arrive at the above formulae for $\rho^\lambda(g)$.

Next assume that g has exactly k_i cycles of length i , $i = 1, 2, \dots$ in its decomposition into disjoint cycles. We will write $g = (1^{k_1} 2^{k_2} \dots)$ in this case. Then $\alpha(g) = k_1 - 1$, $\alpha(g^2) = k_1 + 2k_2 - 1$, and $\alpha(g^3) = k_1 + 3k_3 - 1$. Let $\rho = \rho^\lambda$ for short. We also assume that $g \neq 1$ nor g is a 2-cycle; in particular, $-1 \leq \alpha(g) \leq n-4$ and $-1 \leq \alpha(g^2) \leq n-1$.

Consider the case $\lambda = (n-2, 1^2)$. Then $\rho(\mathbf{t}) = (n^2 - 7n + 10)/2$, and

$$1 - n \leq -\alpha(g^2) \leq 2\rho(g) = \alpha(g)^2 - \alpha(g^2) \leq (n-4)^2 + 1 \leq 2\rho(\mathbf{t}).$$

Next assume that $\lambda = (n-2, 2)$. Then $\rho(\mathbf{t}) = (n^2 - 7n + 12)/2$. Furthermore,
 $-1 - 2n \leq 2\rho(g) = \alpha(g)(\alpha(g) - 2) + \alpha(g^2) - 2 \leq (n-4)(n-6) + n - 3 \leq 2\rho(\mathbf{t})$.

Now we consider the case $\lambda = (n-3, 3)$. Then $\rho(\mathbf{t}) = (n-3)(n-4)(n-5)/6$, and

$$\rho(g) = (k_1 - 1)(k_1 - 2)(k_1 - 3)/6 + (k_1 - 1)(k_2 - 1) + k_3.$$

The desired estimate is clear if $k_1 = 0$. Assume that $k_1, k_2 \geq 1$, in particular $\rho(g) \geq 0$. Since $\rho(g)$ is increasing when we replace $(1^{k_1} 2^{k_2} 3^{k_3})$ by $(1^{k_1+k_3} 2^{k_2+k_3})$, we may assume that $k_3 = 0$. Also, since $\rho(g)$ is increasing when we replace $(1^{k_1} 2^{k_2})$ by $(1^{k_1+2} 2^{k_2-1})$ for $k_2 \geq 2$, we may assume that $k_2 = 1$. It follows that $\rho(g)$ is maximized when g is a 2-cycle. Finally, let $k_2 = 0$. Again the desired estimate is clear if $1 \leq k_1 \leq 5$, so we may assume $k_1 \geq 6$ and $k_j \geq 1$ for some $j \geq 3$; in particular, $\rho(g) \geq 0$. Notice that $\rho(g)$ increases when we replace a j -cycle by $(1^{j-3} 3^1)$ for $j \geq 4$, and when we replace $(1^{k_1} 3^{k_3})$ by $(1^{k_1+3} 3^{k_3-1})$ for $k_3 \geq 2$. Hence $\rho(g) \leq \rho(3\text{-cycle}) \leq \rho(\mathbf{t})$.

Next assume that $\lambda = (n-3, 1^3)$. Then $\rho(\mathbf{t}) = (n-2)(n-3)(n-7)/6$. The desired estimate is clear if $\alpha(g) \leq 1$ or if $n = 9$. On the other hand, if $2 \leq \alpha(g) \leq n-5$ and $n \geq 10$, then

$$6|\rho(g)| = |\alpha(g)^3 - 3\alpha(g)\alpha(g^2) + 2\alpha(g^3)| \leq (n-5)^3 + 3(n-5) + 2(n-1) \leq 6\rho(\mathbf{t}).$$

Also, if $\alpha(g) = n-4$ and $n \geq 10$, then $6\rho(g) = (n-4)^2(n-7) + 2(n-1) < 6\rho(\mathbf{t})$.

Finally, we consider the case $\lambda = (n-3, 2, 1)$. Then $\rho(\mathbf{t}) = (n-2)(n-4)(n-6)/3$, and

$$3\rho(g) = (k_1 - 1)^3 - 3(k_1 - 1)^2 - (k_1 - 1) - 3(k_3 - 1).$$

The desired estimate is clear if $k_1 \leq 4$, so we may assume $k_1 \geq 5$. Observe that $\rho(g)$ increases when we replace a j -cycle by $(1^{j-2} 2^1)$ for $j \geq 4$, or if we replace a 3-cycle by $(1^1 2^1)$ for $k_3 \geq 1$, or if we replace a 2-cycle by (1^2) for $k_2 \geq 2$. It now readily follows that $|\rho(g)| \leq \rho(\mathbf{t})$. \square

Proposition 3.4. *Let G be as in (\clubsuit) and $S = A_n$ for some $n \geq 8$. Then $\chi(1) = n-1$, $L = A_n$, and L acts on V as on its deleted natural permutation module. Moreover, one of the following holds.*

(i) $\text{age}^*(g) = 1/2$, $\Delta(g) = 2$, and a scalar multiple of g is a 2-cycle, acting on V as a reflection.

(ii) $\text{age}^*(g) = 1$, $\Delta(g) = 3$ or 4 , and a scalar multiple of g is a 3-cycle, or a double transposition, both acting on V as a (complex) bireflection.

Proof. 1) First we consider the case $L = 2A_n$. Since $\text{Aut}(A_n) = S_n$ and $C_G(L/Z(L)) = Z(G)$, we may replace G by $H \in \{2A_n, 2S_n\}$. By Lemma 2.23 and (4) we have $4.556 > \Delta(g) \geq \chi(1)/8$ and so $\chi(1) \leq 36$. It is well known (cf. e.g. [KT]) that $\chi(1) \geq 2^{\lfloor n/2 \rfloor - 1}$, hence $n \leq 13$. Now we can go through the irreducible spin characters

of H for $8 \leq n \leq 13$ as listed in [Atlas] and check that $\Delta(g)$ can be less than 4.556 only when $\chi(1) = 8$, $n = 8$ or 9 , and $\Delta(g) = 4$. However, in this exceptional case, $\text{age}^*(g) > 1$.

2) Next we assume that $L = \mathbf{A}_n$ and moreover $\chi|_L$ is not the character of the deleted natural permutation module. Again as above we may replace G by $H \in \{\mathbf{A}_n, \mathbf{S}_n\}$. By Proposition 2.22(ii), $|\chi(g)/\chi(1)| \leq 1 - 1/2n$, whence $4.556 > \Delta(g) \geq \chi(1)/2n$ and $\chi(1) < (9.112)n$. Also we choose $\lambda \vdash n$ such that $\chi|_L$ is an irreducible constituent of $\rho^\lambda|_L$. By our assumption, $\lambda \notin R_n(1)$.

Consider the case $n \geq 14$. Then by Lemma 3.2(i) (and by [GAP] for $n = 14$), either $\rho^\lambda(1) \geq n(n-1)(n-5)/6$, or $\lambda \in R_n(2)$. Since $\chi(1) \geq \rho^\lambda(1)/2$, in the former case we would have $\chi(1) \geq (9.75)n$, a contradiction. Hence $\lambda \in R_n(2)$; in particular, $\chi|_L = \rho^\lambda|_L$. But in this case, Lemma 3.3 and its proof imply that $\Delta(g) \geq \Delta(\mathbf{t}) \geq 2n - 6 \geq 22$, again a contradiction.

Finally, let $8 \leq n \leq 13$. An inspection of irreducible characters of H [Atlas] reveals that $\Delta(g) > 4.556$ in all cases.

3) We have shown that $\chi(1) = n - 1$ and $\chi|_L$ is the character of the deleted natural permutation module. We may write $g = \alpha h$, where $h \in \mathbf{S}_n$ and $\alpha \in \mathbb{C}^\times$. Then $|\chi(g)| = |\chi(h)| = |\mu(h) - 1|$, where $\mu(h)$ is the number of points fixed by the permutation h . Since $\Delta(g) < 4.556$ and $n \geq 8$, we see that $n - 2 \geq \mu(h) \geq n - 4$. If $\mu(h) = n - 2$, then h is a 2-cycle, $\Delta(g) = 2$ and $\text{age}^*(g) = \text{age}(h) = 1/2$. If $\mu(h) = n - 3$, then h is a 3-cycle, $\Delta(g) = 3$ and $\text{age}^*(g) = \text{age}(h) = 1$. If $\mu(h) = n - 4$, then $\Delta(g) = 4$ and h is either a double transposition, or a 4-cycle. In the former case $\text{age}^*(g) = \text{age}(h) = 1$. In the latter case $\Delta(g) = 4$ and $\text{age}^*(g) > 1$ by Lemma 2.21(iii). \square

From now on we may assume that $S \not\cong \mathbf{A}_n$ for any $n \geq 8$. By Lemma 2.19(i) and (4), there is some $h \in L \setminus Z(L)$ with

$$(5) \quad \Delta(h) \leq 4\Delta(g) < 18.224,$$

which implies by Proposition 2.22(i) that $\chi(1)/20 < 18.224$ and so

$$(6) \quad \chi(1) \leq 364.$$

Let $\mathfrak{d}(S)$ denote the smallest degree of a projective complex irreducible representation of S . We will freely use the precise value of $\mathfrak{d}(S)$ as recorded in [T].

3.3. Classical groups. To handle the finite classical groups, we will also need to estimate character ratios for their *Weil representations* (cf. [TZ2], [GMST] and references therein, for definitions and detailed information on Weil representations).

Lemma 3.5. *Let χ be an irreducible complex Weil character of $L = SL_n(q)$ or $SU_n(q)$, $n \geq 3$, $(n, q) \neq (3, 2), (3, 3), (4, 2)$, and let $g \in L \setminus Z(L)$. Then*

$$\frac{|\chi(g)|}{\chi(1)} < \frac{q^{n-1} + q^2}{q^n - q} \leq \frac{2}{3}.$$

Proof. First we consider the case $L = SU_n(q)$ and let $\mathcal{N} = \mathbb{F}_{q^2}^n$ denote the natural module for L . Fix a primitive $(q+1)^{\text{th}}$ -root δ of unity in \mathbb{F}_{q^2} , and let d_k denote the dimension of the subspace $\text{Ker}(g - \delta^k \cdot \text{Id})$ of \mathcal{N} , for $0 \leq k \leq q$. Then the explicit formula for χ as given in [TZ2] implies that $\chi(1) \geq (q^n - q)/(q+1)$ and $(q+1)|\chi(g)| \leq S := \sum_{k=0}^q q^{d_k}$. Clearly, $\sum_{k=0}^q d_k \leq n$ and $0 \leq d_k \leq n-1$. Without loss we may assume that $d_1 = \max_{0 \leq k \leq q} d_k$. Now $S \leq (q+1)q^{n-3} < q^{n-1}$ if $d_1 \leq n-3$, $S \leq q^{n-2} + q^2 + q - 1 < q^{n-1} + q^2$ if $d_1 = n-2$, and $S \leq q^{n-1} + 2q - 1 < q^{n-1} + q^2$ if $d_1 = n-1$, and so we are done.

Next, let $L = SL_n(q)$ and let $\mathcal{N} = \mathbb{F}_q^n$ denote the natural module for L . Fix a primitive $(q-1)^{\text{th}}$ -root ϵ of unity in \mathbb{F}_q , and let e_k denote the dimension of the subspace $\text{Ker}(g - \epsilon^k \cdot \text{Id})$ of \mathcal{N} , for $0 \leq k \leq q-2$. Then the explicit formula for χ as given in [TZ2] implies that $\chi(1) \geq (q^n - q)/(q-1)$ and $(q-1)|\chi(g)| \leq R := \sum_{k=0}^{q-2} q^{e_k} + 2q - 2$. Clearly, $\sum_{k=0}^{q-2} e_k \leq n$ and $0 \leq e_k \leq n-1$. Without loss we may assume that $e_1 = \max_{0 \leq k \leq q-2} e_k$. Now $R \leq (q-1)(q^{n-3} + 2) < q^{n-1} + q^2$ if $e_1 \leq n-3$, $R \leq q^{n-2} + q^2 + 3q - 5 < q^{n-1} + q^2$ if $e_1 = n-2$, and $R \leq q^{n-1} + 4q - 5 < q^{n-1} + q^2$ if $e_1 = n-1$, and so we are again done. \square

Lemma 3.6. *Let χ be an irreducible complex Weil character of $L = Sp_{2n}(q)$, q odd, $n \geq 2$, and let $g \in L \setminus Z(L)$. Then*

$$\frac{|\chi(g)|}{\chi(1)} \leq \begin{cases} \frac{q^{n-1/2} + 1}{q^n + 1}, & q \equiv 1 \pmod{4} \text{ and } \pm g \text{ is a transvection,} \\ \frac{(q^{2n-1} + 1)^{1/2}}{q^n - 1}, & q \equiv 3 \pmod{4} \text{ and } \pm g \text{ is a transvection,} \\ \frac{2q^{n-1}}{q^n - 1}, & \pm g \text{ is not a transvection.} \end{cases}$$

In particular, $|\chi(g)/\chi(1)| \leq 0.675$ unless $(n, q) = (3, 3), (2, 3)$.

Proof. Note that $\chi(1) = (q^n - \epsilon)/2$ for some $\epsilon = \pm 1$. If $\pm g$ is a transvection in L then by [TZ2], $|\chi(g)| = (q^{n-1/2} - \epsilon)/2$ when $q \equiv 1 \pmod{4}$, and $|\chi(g)| = \sqrt{q^{2n-1} + 1}/2$ when $q \equiv 3 \pmod{4}$. Assume $\pm g$ is not a transvection, i.e. the subspace $\text{Ker}(g \pm \text{Id})$ on the natural module \mathbb{F}_q^{2n} of L has dimension at most $2n-2$. Also consider the reducible Weil character ω of L (that has χ as one of its irreducible constituents), cf. [GMST]. This character arises from the action of L as an outer automorphism subgroup of the extraspecial p -group of order p^{1+2nf} and exponent p , where $q = p^f$ and p is prime. By [GT3, Proposition 2.8], $|\omega(g)|, |\omega(-g)| \leq q^{n-1}$. One can write $\omega = \chi + \eta$ for another irreducible Weil character η of L , and moreover, $|\omega(-g)| = |\chi(g) - \eta(g)|$. It follows that $|\chi(g)| \leq q^{n-1}$. \square

3.3.1. $S = PSL_n(q)$, $n \geq 3$, $(n, q) \neq (3, q \leq 7), (4, 3), (5, 2)$. Under these assumptions, $\mathfrak{d}(S) = (q^n - q)/(q - 1)$. Hence (6) implies that $3 \leq n \leq 8$; moreover, $q = 2$ if $n = 7, 8$, $q \leq 3$ if $n = 6$, $q \leq 4$ if $n = 5$, $q \leq 5$ if $n = 4$, and $q \leq 17$ if $n = 3$.

In fact, if in addition $\chi|_L$ is a Weil representation, then $18.224 > \Delta(h) \geq \chi(1)/3$ by Lemma 3.5, and so instead of (6) we have the much stronger upper bound $\chi(1) \leq 54$. Now in the cases $(n, q) = (8, 2), (7, 2), (6, 3), (5, 4), (5, 3)$, and $(3, q \geq 8)$, the upper bound (6) and [TZ1, Theorem 3.1] imply that $\chi|_L$ is indeed a Weil representation, of degree at least 72, giving a contradiction. Also, the case $(n, q) = (4, 2)$ has already been considered in Proposition 3.4.

Assume $(n, q) = (6, 2)$ or $(4, 4)$. Then $L = SL_n(q)$, and its character table is available in [GAP]. It is straightforward to check that there is no nontrivial $\chi \in \text{Irr}(L)$ and $h \in L \setminus Z(L)$ with $\Delta(h) < 18.224$ (notice that we need to check only the non-Weil characters of degree at most 364).

It remains to analyze the case $S = PSL_4(5)$. The character degrees of $SL_4(5)$ are listed by F. Lübeck [Lu2]. In particular, we see that all the non-trivial irreducible characters of $R := SL_4(5)$ have degree 155, 156 (and they are Weil characters in these two cases), 248 (and there are exactly two characters of this degree), or at least 403. Hence we may assume that $\chi(1) = 248$. An inspection of character degrees as listed in [St] shows that $GL_4(5)$ has no irreducible characters of degree 248. Thus $\chi|_R$ is not stable under $GL_4(5)$. Since $\text{Out}(R)$ is a dihedral group of order 8 and $GL_4(5)$ induces the unique cyclic subgroup of order 4 of $\text{Out}(R)$, it follows that the inertia group of $\chi|_R$ in $\text{Out}(R)$ is an elementary abelian 2-group. But $\chi|_R$ extends to G . Thus G can induce only an elementary abelian 2-subgroup of $\text{Out}(R)$. We conclude that $g^2 \in Z(G)L$. Notice that $\chi(1) - |\chi(v)| \geq \chi(1)/20 = 12.4$ for any $v \in L \setminus Z(L)$ by Proposition 2.22(i). Together with (5), this implies that $4\Delta(g) \geq \Delta(h) \geq 12.4$ and so $\Delta(g) \geq 3.1$. We will complete the case $S = PSL_4(5)$ by showing that $\text{age}^*(g) > 1$.

First we suppose that $g^2 \notin Z(G)$. Then $\Delta(g^2) \geq 12.4$ as above, and so $\text{age}^*(g^2) \geq 24.8/(2.9\pi) > 2.72$ by Proposition 2.17(iii). It now follows by Lemma 2.2(v) that $\text{age}^*(g) > \text{age}^*(g^2)/2 > 1.36$. Finally, assume that $g^2 \in Z(G)$. Then g acts on a

suitable basis of V via the matrix $\alpha \cdot \text{diag} \left(\underbrace{1, \dots, 1}_k, \underbrace{-1, \dots, -1}_l \right)$ for some $\alpha \in \mathbb{C}^\times$

and $1 \leq k, l < k + l = 248$. It is shown in [GS] that $\alpha(gZ(G)) \leq 6$, whence $k, l \geq 42$ by Lemma 2.16. It follows that $|\chi(g)| = |248 - 2l| \leq 164$, $\Delta(g) \geq 84$, and $\text{age}^*(g) \geq 168/(2.9\pi) > 18$ again by Proposition 2.17(iii).

3.3.2. $S = PSU_n(q)$, $n \geq 3$, $(n, q) \neq (3, q \leq 8), (4, 2), (4, 3), (5, 2), (6, 2)$. Under these assumptions, $\mathfrak{d}(S) = (q^n - q)/(q + 1)$ if n is odd and $(q^n - 1)/(q + 1)$ if $2|n$. Hence (6) implies that $3 \leq n \leq 10$; moreover, $q = 2$ if $7 \leq n \leq 10$, $q \leq 3$ if $n = 6$, $q \leq 4$ if $n = 5$, $q \leq 7$ if $n = 4$, and $q \leq 19$ if $n = 3$. As in §3.3.1, if in addition $\chi|_L$ is a Weil representation, then instead of (6) we have the much stronger upper bound $\chi(1) \leq 54$ (in fact $\chi(1) \leq 39$ if $(n, q) = (7, 2)$ or $(4, 4)$). Now in the cases $(n, q) = (10, 2), (9, 2), (8, 2), (6, 3), (5, 4), (5, 3), (4, 7)$, and $(3, q \geq 9)$, the upper bound (6) and [TZ1, Theorem 4.1] imply that $\chi|_L$ is indeed a

Weil representation, of degree at least 60, giving a contradiction. The same argument applies to $(n, q) = (7, 2)$ as the Weil representations of $SU_7(2)$ have degree at least 42 and the non-Weil representations have degree at least 860.

Assume $(n, q) = (4, 4)$. Then $L = SU_4(4)$, and its character table is available in [GAP]. It is straightforward to check that there is no nontrivial $\chi \in \text{Irr}(L)$ and $h \in L \setminus Z(L)$ with $\Delta(h) < 18.224$ (notice that we need to check only the non-Weil characters of degree at most 364).

It remains to analyze the case $S = PSU_4(5)$. The character degrees of $SU_4(5)$ are listed by F. Lübeck [Lu2]. In particular, we see that all the non-trivial irreducible characters of $SU_4(5)$ have degree 104, 105 (and they are Weil characters in these two cases), 273 (and there are exactly two characters of this degree), or at least 378. Hence we may assume that $\chi(1) = 273$. Checking the character table of $PSU_4(5)$ (available in [GAP]), we see that it also has exactly two irreducible characters of degree 273. It follows that $L = S = PSU_4(5)$. Direct inspection of these two characters of S reveals that $\Delta(h) \geq 250$, a contradiction.

3.3.3. $S = PSp_{2n}(q)$, $n \geq 2$, $(n, q) \neq (2, q \leq 5), (3, 2), (3, 3), (4, 2)$. Under these assumptions, $\mathfrak{d}(S) = (q^n - 1)/2$ if q is odd and $(q^n - 1)(q^n - q)/2(q + 1)$ if $2|q$. Hence (6) implies that $2 \leq n \leq 6$; moreover, $q = 3$ if $n = 6$, $q \leq 3$ if $n = 5$, $q = 3, 5$ if $n = 4$, $q = 5, 7, 9$ if $n = 3$; if $n = 2$ then either $q \leq 27$ and q odd or $q = 8$. Moreover, if in addition q is odd and $\chi|_L$ is a Weil representation, then, by Lemma 3.6, instead of (6) we have the much stronger upper bound $\chi(1) \leq 56$ (in fact $\chi(1) \leq 29$ if $(n, q) = (2, 7)$ or $(2, 9)$). Now in the cases $(n, q) = (6, 3), (5, 3), (4, 5), (3, 5), (3, 7), (3, 9)$, and $(2, q \geq 11)$, the upper bound (6) and [TZ1, Theorem 5.2] imply that $\chi|_L$ is indeed a Weil representation, of degree at least 60, giving a contradiction.

Assume $(n, q) = (2, 7)$ or $(2, 9)$. The character table of $Sp_{2n}(q)$ is determined in [Sr]. It is now straightforward to check that $\Delta(h) \geq 100$ if $\chi|_L$ is a non-Weil character of degree at most 364. Moreover, the Weil characters of $Sp_4(9)$ have degree 40 or 41, larger than the bound 29 mentioned above. On the other hand, when $(n, q) = (2, 7)$, none of the Weil characters (of degree 24 or 25) is fixed by an outer automorphism of $Sp_4(7)$. This implies that $G = Z(G)L$ and so we may assume $g \in L$ in this case. Hence, if $\chi|_L$ is a Weil character, then $4.556 > \Delta(g) \geq (1 - 0.675)\chi(1)$ by Lemma 3.6, and so $\chi(1) \leq 14$, a contradiction. The same argument excludes the Weil characters of $Sp_8(3)$; all other nontrivial irreducible characters of $Sp_8(3)$ have degree at least 780 by [TZ1, Theorem 5.2], hence we are done in the case $(n, q) = (4, 3)$. If $(n, q) = (5, 2)$, then $\text{Out}(L) = 1$ and so we may assume that $g \in L$, whence $\Delta(g) \geq \chi(1)/20 \geq 7.75$ as $\mathfrak{d}(S) = 155$. Finally, inspecting the character table of $Sp_4(8)$ (available in [GAP]), we see that $\Delta(h) \geq 168$, again a contradiction when $(n, q) = (2, 8)$.

3.3.4. $S = P\Omega_n^\epsilon(q)$, $n \geq 7$, $(n, q) \neq (7, 3), (8, 2), (10, 2)$. If $(n, q) \neq (8, 3)$ in addition, then $\mathfrak{d}(S) \geq 620$ by [TZ1], and so we are done. Consider the case $S =$

$P\Omega_8^\pm(3)$. Notice that $Spin_7(3)$ embeds in $Spin_8^\pm(3)$ and any faithful irreducible character of $Spin_7(3)$ has degree at least 520. Hence the bound (6) implies that $L = S$ (this can also be deduced using the list of character degrees of $Spin_8^\pm(q)$ as given in [Lu2]). Inspecting the character table of $P\Omega_8^\pm(3)$ (available in [Atlas]), we see that $\Delta(h) \geq 189$, a contradiction.

3.3.5. $S = PSL_2(q)$, $q \geq 37$. In these cases, $\chi(1) \geq (q-1)/\gcd(2, q-1)$ and $|\chi(h)| \leq (\sqrt{q}+1)/2$, cf. [D]. In particular, $|\chi(h)/\chi(1)| \leq 1/(\sqrt{q}-1) < 0.2$, and so (5) implies that $\chi(1) \leq 22$. Since we are assuming $q \geq 37$, this in turn forces that $q = 37, 41$, or 43 , and $\chi|_L$ is in fact a Weil character. But for these values of q , none of the Weil characters of L is fixed by an outer automorphism of L . Hence $G = Z(G)L$, and so we may assume that $g \in L$. Thus $|\chi(g)/\chi(1)| < 0.2$ as above, and $\Delta(g) > (0.8)\chi(1) \geq 14.4$, a contradiction.

3.4. Exceptional groups of Lie type. Let S be a simple exceptional group of Lie type. If S is not isomorphic to ${}^2B_2(q)$ with $q \leq 32$, $G_2(q)$ with $q \leq 7$, ${}^3D_4(q)$ with $q \leq 3$, ${}^2F_4(2)'$, or $F_4(2)$, then $\mathfrak{d}(S) \geq 504$, see e.g. [Lu1]. Consider the case $S = G_2(7)$. Then L has a unique nontrivial irreducible character of degree at most 364 (namely 344), and this character is labeled as χ_{32} in the generic character table of $G_2(q)$ [H]. One can now check that $\Delta(h) \geq 332$ for $\chi|_L = \chi_{32}$. Similarly, if $S = {}^3D_4(3)$ then L has a unique nontrivial irreducible character of degree at most 364 (namely 219). This character is unipotent, and its values are computed in [Sp]. In particular, one can check that $\Delta(h) \geq 195$ in this case.

3.5. Small groups. The list of our “small” groups consists of all the finite simple groups not considered in the above subsections, that is: A_n with $5 \leq n \leq 7$, $PSL_2(q)$ with $7 \leq q \leq 32$, $PSL_3(q)$ with $3 \leq q \leq 7$, $PSL_4(3)$, $SL_5(2)$, $PSU_3(q)$ with $3 \leq q \leq 8$, $SU_4(2)$, $PSU_4(3)$, $SU_5(2)$, $PSU_6(2)$, $Sp_4(4)$, $PSp_4(5)$, $Sp_6(2)$, $PSp_6(3)$, $Sp_8(2)$, $\Omega_7(3)$, $\Omega_8^\pm(2)$, $\Omega_{10}^\pm(2)$, ${}^2B_2(q)$ with $8 \leq q \leq 32$, $G_2(q)$ with $3 \leq q \leq 5$, ${}^3D_4(2)$, ${}^3F_4(2)'$, $F_4(2)$, and 26 sporadic simple groups. Notice that the character table of the universal cover of S is known (see [GAP]) in all these cases.

Recall we are assuming that $L = G^{(\infty)}$ is quasi-simple, and $\chi \in \text{Irr}(G)$ is irreducible over L ; moreover, and $1 < \chi(1) \leq 364$ by (6). The last condition excludes the cases $S \in \{J_4, Fi_{23}, Fi'_{24}, Ly, BM = F_2, M = F_1\}$. We will use the character tables of the universal cover of S as given in [Atlas], as well as the notation therein for the conjugacy classes in $G/Z(G)$.

3.5.1. Sporadic groups. One can check that

- $\Delta(g) \geq 6$ if $S = M_{22}, Suz$;
- $\Delta(g) \geq 8$ if $S = M_{11}, M_{12}$, or if $S = J_2$ but $\chi(1) > 6$; and
- $\Delta(g) \geq 12$ if $S = M_{23}, M_{24}, J_1, J_3, HS, McL, He, Ru, HN, Fi_{22}, Co_3, Co_2, Co_1, O'N, Th$

for all χ satisfying the above hypotheses.

Assume that $S = J_2$ and $\chi(1) = 6$; in particular, $L = 2 \cdot J_2$ and $G = Z(G)L$. Then one can check that $\Delta(g) \geq 5 - \sqrt{5} > 8 - 4\sqrt{2}$. Next, suppose that $0 < \text{age}^*(g) \leq 1$; in particular, $|\chi(g)| > 1.444$. Then, in the notation of [Atlas], we may assume that $\chi|_L = \chi_{22}$, and the class of $gZ(G)$ in S is one of the following: $2A$, $3A$, $4A$, $5B$, $5C$, $10D$, and $15B$. The first two cases lead to the row of $2 \cdot J_2$ in Table I. In the last two cases, in the notation of Corollary 2.18 we have $\delta \geq 6\pi/5$, but $\Delta(g) > 4.38$, whence $\text{age}^*(g) > 1$ by Corollary 2.18. In the case of class $4A$, a multiple of g has spectrum $1, 1, i, i, -i, -i$, and so $\text{age}^*(g) > 1$ by Lemma 2.21 (with $i = \sqrt{-1}$). Finally, in the case of classes $5B$ and $5C$, none of the eigenvalue of g occurs with multiplicity ≥ 3 , and so $\text{age}^*(g) > 1$ by Lemma 2.21.

3.5.2. Small alternating groups: $S = A_n$ with $5 \leq n \leq 7$. Arguing as in the proof of Proposition 3.4 (and using Lemmas 2.20 and 2.21), we may assume that $\chi|_L$ is not the character of the deleted natural permutation module. First we consider the case $S = A_5$. Direct check using [Atlas] shows that $\Delta(g) \geq (3 - \sqrt{5})/2$ if $d = 2$ and $\Delta(g) \geq (5 - \sqrt{5})/2 > 3 - \sqrt{3}$ if $d = 3$. Assume $d = 4$ (and so $L = 2 \cdot A_5$ by our assumptions). If $gZ(G)$ belongs to the class $5A$ or $5B$ of $G/Z(G)$, then $\Delta(g) = 3$ and $\text{age}^*(g) > 1$ by Lemma 2.21(iv). If $gZ(G)$ belongs to the class $4A$, then $\Delta(g) = 4$ and $\text{age}^*(g) > 1$ by Lemma 2.20 (with $m = 2$). Similar arguments apply to the case $d = 5$. If $d = 6$, then $\Delta(g) \geq 6 - \sqrt{2} > 4.556$.

Assume $n = 6$. Then the assumptions on χ , L , and $\Delta(g)$ lead to one of the following three possibilities.

- $d = 3$, $L = 3A_6$, and $\Delta(g) \geq (5 - \sqrt{5})/2 > 3 - \sqrt{3}$.
- $d = 4$, $L = 2 \cdot A_6$, and $\Delta(g) \geq 2$. The classes $2A$, $2B$, $2C$, $3A$, $3B$, and $6B$ lead to three rows of Table I. The other classes are excluded by Lemmas 2.20 and 2.21.
- $d = 6$, $L = 3 \cdot A_6$, $\Delta(g) = 4$, and $gZ(G)$ belongs to the class $2A$, which leads to a row in Table I.

Assume $S = A_7$. Then the assumptions on χ , L , and $\Delta(g)$ lead to one of the following two possibilities.

- $d = 4$, $L = 2 \cdot A_7$, $G = Z(G)L$, and $\Delta(g) \geq 2$. The classes $2A$, $3A$, $3B$, and $7A$ lead to two rows of Table I. The other classes are excluded by Lemmas 2.20 and 2.21.
- $d = 6$, $L = 3 \cdot A_7$, $G = Z(G)L$, $\Delta(g) = 4$, and $gZ(G)$ belongs to the classes $2A$ or $6A$. The former case leads to a row in Table I, and $\text{age}^*(g) > 1$ in the latter case by Lemma 2.20 (with $m = 2$).

Lemma 3.7. *There are subgroups $G = C_3 \times 2A_m < GL_4(\mathbb{C})$ with $m = 6, 7$ which give a basic non-RT pair not of reflection type. This pair is of AV-type if $m = 6$.*

Proof. The faithful representation of G on $V = \mathbb{C}^4$ gives rise to a unique conjugacy class g^G of non-central elements of age < 1 , namely class $3A$ in $G/Z(G) \simeq A_m$. Let $K := \langle g^G \rangle$. Then $Z(G)K = G$ by simplicity of $G/Z(G)$, but $g \notin [G, G] = 2A_m$. It follows that $K = G$ and so G gives a basic non-RT pair. Furthermore, $Z(GL(V))G$

does not contain any complex reflection, hence this pair is not of reflection type. Finally, if $m = 6$ then the representation of $G = C_3 \times SL_2(9) < C_3 \times Sp_4(3)$ on V can be written over $\mathbb{Q}(\sqrt{-3})$ and so the corresponding basic non-RT pair is of AV-type. \square

3.5.3. Small finite groups of Lie type. Let S be any of the small simple finite groups of Lie type listed at the beginning of §3.5. Using [Atlas], it is straightforward to check that $\Delta(g) \geq 5$ for all characters χ satisfying the above hypotheses, except possibly for one of the following cases. (Note that, it suffices to consider only subgroups of G that induce *cyclic extensions* of S in $\text{Aut}(S)$, since $\langle g, L \rangle$ is such a subgroup.)

- $d = 8$, $L = 2 \cdot \Omega_8^+(2)$. Here, either $\Delta(g) = 2$, $gZ(G)$ belongs to class $2F$, and g acts as a reflection, or $\Delta(g) \geq 3$. In the latter case, either we get the row of $\Omega_8^+(2)$ in Table I with complex bireflections of order 2 and 3, or $\text{age}^*(g) > 1$ by Lemma 2.21 (when $gZ(G)$ has order ≤ 5) and Corollary 2.18 (with $\delta \geq 4\pi/3$).

- $d = 7$ or 8 , and $S = Sp_6(2)$. If $d = 8$, then $\Delta(g) \geq 4$ and $\text{age}^*(g) > 1$. Assume $d = 7$. Then either $\Delta(g) = 2$, $gZ(G)$ belongs to class $2F$, and $-g$ acts as a reflection, or $\Delta(g) \geq 3$. In the latter case, either we get the row of $Sp_6(2)$ in Table I with complex bireflections of order 2 and 3, or $\text{age}^*(g) > 1$ by Lemma 2.21.

- $d = 10$, $L = SU_5(2)$, $\Delta(g) = 4$, $gZ(G)$ belongs to class $2A$, and g acts as a bireflection.

- $d = 6, 20$, or 21 , and $S = PSU_4(3)$. Assume $d = 6$. Then either $\Delta(g) = 2$, $gZ(G)$ belongs to class $2D$, and g acts as a reflection, or $\Delta(g) \geq 3$. In the latter case, either we get the row of $PSU_4(3)$ in Table I with complex bireflections of order 2 and 3 and an element with spectrum $(1, 1, 1, e^{2\pi/3}, e^{2\pi/3}, e^{2\pi/3})$, or $\text{age}^*(g) > 1$ by Lemma 2.21 and Corollary 2.18 (with $\delta \geq 4\pi/3$) (and a direct check for some elements of order 6). If $d = 21$ and $L = S$, then $\Delta(g) \geq 12$. In all the remaining cases, $\Delta(h) \geq 13$ for all $h \in L \setminus Z(G)$, and so $\Delta(g) \geq 13/4$ by Lemma 2.19. We claim that we also have $\text{age}^*(g) > 1$. Assume the contrary: $\text{age}^*(g) \leq 1$. Let K be any subgroup of G that contains L and induces a subgroup C_2 of $\text{Out}(S) = D_8$ while acting on L . It is straightforward to check that, for any $h \in K \setminus Z(K)$, $\Delta(h) \geq 10$ and so $\text{age}^*(h) > 2.19$ by Corollary 2.18. Notice that $\text{Out}(S) = D_8$ has exponent 4. Hence, if $g^2 \notin Z(G)$, we have $g^2 \in K \setminus Z(K)$ for a subgroup K of the aforementioned type, and so $\text{age}^*(g^2) > 2.19$ and $\text{age}^*(g) > 1.095$ by Lemma 2.2(v). Thus $g^2 \in Z(G)$, and so modulo scalars we may assume that g has two eigenvalues 1, resp. -1 , with multiplicity m , resp. $d - m$. By [GS], $\alpha(g) \leq 6$ and so $m, d - m \leq d - 4$ by Lemma 2.16. It follows that $|\chi(g)| = |2m - d| \leq d - 8$, whence $\Delta(g) \geq 8$ and $\text{age}^*(g) > 1$.

- $d = 4, 5$, or 6 , and $S = SU_4(2) \simeq PSp_4(3)$. Assume $d = 6$. Then either $\Delta(g) = 2$, $gZ(G)$ belongs to class $2C$, and g acts as a reflection, or $\Delta(g) \geq 3$. In the latter case, either we get a row with $(d, L) = (6, SU_4(2))$ in Table I, or $\text{age}^*(g) > 1$ by Lemma 2.21 and Corollary 2.18 (with $\delta \geq 4\pi/3$). Assume $d = 5$. Then either $\Delta(g) = 2$,

$gZ(G)$ belongs to class $2A$, and $-g$ acts as a reflection, or $\Delta(g) \geq 5 - \sqrt{7} > 8 - 4\sqrt{2}$. In the latter case, either we get two rows with $(d, L) = (5, SU_4(2))$ in Table I, or $\text{age}^*(g) > 1$ by Lemma 2.21 and Corollary 2.18 (with $\delta \geq 4\pi/3$) (and a direct check for some elements of order 6). Finally, assume $d = 4$, and so $L = Sp_4(3)$. This case by far has the most (twelve) classes of elements g with $0 < \text{age}^*(g) \leq 1$ (leading to two rows in Table I), and is handled by a direct case-by-case argument. In this case we always have $\Delta(g) \geq 4 - \sqrt{7}$.

- $d = 6$ or 7 , and $S = SU_3(3)$. Here we have $\Delta(g) \geq 3$, and, aside from the entries with $(d, L) = (6, SU_3(3))$ and $(7, SU_3(3))$ in Table I, $\text{age}^*(g) > 1$ by Lemmas 2.20, 2.21, and Corollary 2.18 (with $\delta \geq 4\pi/3$).

- $d = 6$ and $L = 6 \cdot PSL_3(4)$. Here, $\Delta(g) \geq 4$, and either we are in the row of $(d, L) = (6, 6 \cdot PSL_3(4))$ in Table I, or $\text{age}^*(g) > 1$.

- $d = 6$ or 7 , and $S = PSL_2(13)$. If $d = 7$, then $\Delta(g) > 4.69$. If $d = 6$, then either $\Delta(g) \geq 5$, or $\Delta(g) > 3.69$ and $\text{age}^*(g) > 1$ by Corollary 2.18 (with $\delta = 16\pi/13$).

- $d = 5$ or 6 , and $S = PSL_2(11)$. If $d = 6$, then either $\Delta(g) \geq 5$, or $\Delta(g) > 4.26$ and $\text{age}^*(g) > 1$ by Corollary 2.18 (with $\delta = 14\pi/11$). If $d = 5$, then $\Delta(g) \geq 5 - \sqrt{3}$, and either we are in the row of $(d, L) = (5, PSL_2(11))$ in Table I, or $\text{age}^*(g) > 1$ (by direct calculation).

- $d = 3, 4$, or 6 , and $S = PSL_2(7)$. If $d = 3$, then $\Delta(g) \geq 3 - \sqrt{2}$. If $d = 6$, then either we are in the row $(d, L) = (6, PSL_2(7))$ of Table I, or $\Delta(g) > 4$ and $\text{age}^*(g) > 1$ by Corollary 2.18 (with $\delta \geq 5\pi/4$). Finally, if $d = 4$, then $\Delta(g) \geq 4 - \sqrt{2}$, and either we arrive at the row $(d, L) = (4, SL_2(7))$ of Table I, or $\text{age}^*(g) > 1$ (by a direct check).

We have completed the proof of Theorem 1.3, as well as of the following result which we will need later.

Theorem 3.8. *Let $G < GL(V)$ be an almost quasi-simple subgroup such that the G -module V is irreducible, primitive, and tensor indecomposable. Assume that $0 < \text{age}^*(g) \leq 1$ for some $g \in G$, and that $d := \dim(V) \geq 4$. Then $(d, G^{(\infty)}, g, \Delta(g), \text{age}^*(g))$ is as listed in Table I. \square*

In Table I, in the cases where $0 < \text{age}^*(g) < 1$, we indicate a minimal group G containing h with $\text{age}(h) = \text{age}^*(g)$. We also list the conjugacy class of $gZ(G)$ in $G/Z(G)$ using the notation of [Atlas], for one representative of the $\text{Aut}(L)$ -conjugacy class of $\chi|_L$.

TABLE I. Non-central elements of small age in almost quasi-simple groups.

d	$L := G^{(\infty)}$	G	g	$\Delta(g) = d_2(g)^2/2$	$\text{age}^*(g)$
4	$Sp_4(3)$	$L \times 3$	$\begin{cases} 3B \\ 3A, 3C, 6A \end{cases}$	$\begin{cases} 4 - \sqrt{7} \\ 4 - \sqrt{7}, 2, 4 - \sqrt{3} \end{cases}$	$\begin{cases} 1/3 \\ 2/3 \end{cases}$
4	$2 \cdot A_6$	$L \times 3$	$3A$	2	$2/3$
4	$2 \cdot A_7$	$L \times 3$	$3A$	2	$2/3$
4	$2 \cdot A_5$	L	$2A$, resp. $3A$	4, resp. 3	1
4	$2 \cdot A_5$	$L \cdot 2$	$2B$, resp. $6A$	4, resp. $4 - \sqrt{3}$	1
4	$SL_2(7)$	L	$2A$, resp. $3A, 7B$	4, resp. 3, $4 - \sqrt{2}$	1
4	$2 \cdot A_6$	L	$2A$, resp. $3B$	4, resp. 3	1
4	$2 \cdot A_6$	$L \cdot 2$	$2B$, resp. $2C, 6B$	4, resp. 4, $4 - \sqrt{3}$	1
4	$2 \cdot A_7$	L	$2A$, resp. $3B, 7A$	4, resp. 3, 2	1
4	$Sp_4(3)$	L	$\begin{cases} 2AB, 3A, 3D \\ 4A, 6BE, 12A \end{cases}$	$\begin{cases} 4, 4 - \sqrt{7}, 3 \\ 2, 4 - \sqrt{3}, 3 \end{cases}$	1
5	$SU_4(2)$	$L \times 2$	$2A$	2	$1/2$
5	$SU_4(2)$	$L \times 3$	$3A$	$5 - \sqrt{7}$	$2/3$
5	A_5	L	$2A$	4	1
5	A_6	S_6	$(12)(34)(56)$	4	1
5	$PSL_2(11)$	L	$2A$	4	1
5	$SU_4(2)$	L	$2B$, resp. $3D$	4, resp. 3	1
6	$SU_4(2)$	$L \cdot 2$	$2C$	2	$1/2$
6	$6_1 \cdot PSU_4(3)$	$L \cdot 2_2$	$2D$	2	$1/2$
6	$PSL_2(7)$	L	$2A$	4	1
6	$3 \cdot A_6$	L	$2A$	4	1
6	$3 \cdot A_7$	L	$2A$	4	1
6	$6 \cdot PSL_3(4)$	L	$2A$	4	1
6	$SU_3(3)$	L	$2A$, resp. $3A$	4, resp. 3	1
6	$SU_4(2)$	L	$2AB$, resp. $3AB, 3C$	4, resp. 3, 3	1
6	$6_1 \cdot PSU_4(3)$	L	$2A$, resp. $3A, 3B$	4, resp. 3, 3	1
6	$2 \cdot J_2$	L	$2A$, resp. $3B$	4, resp. 3	1
7	$Sp_6(2)$	$L \times 2$	$2A$	2	$1/2$
7	$SU_3(3)$	L	$2A$	4	1
7	$Sp_6(2)$	L	$2C$, resp. $3A$	4, resp. 3	1
8	$2 \cdot \Omega_8^+(2)$	$L \cdot 2$	$2F$	2	$1/2$
8	$2 \cdot \Omega_8^+(2)$	L	$2B$, resp. $3A$	4, resp. 3	1
10	$SU_5(2)$	L	$2A$	4	1
$n-1$	A_n	S_n	2-cycle	2	$1/2$
$n-1$	A_n	L	(123) , resp. $(12)(34)$	3, resp. 4	1

Remark 3.9. *The spectra of elements g with $0 < \text{age}^*(g) \leq 1$ that occur in Table I are listed as follows:*

- (a1) $(-1, 1, \dots, 1)$ (a reflection, $\text{age} = 1/2$);
- (a2) $(e^{2\pi i/3}, 1, 1, 1)$ (a complex reflection, $\text{age} = 1/3$);
- (a3) $(e^{4\pi i/3}, 1, 1, 1)$ (a complex reflection, $\text{age} = 2/3$);
- (a4) $(e^{2\pi i/3}, e^{2\pi i/3}, 1, \dots, 1)$ ($\text{age} = 2/3$);
- (a5) $(e^{\pi i/3}, -1, 1, 1)$ ($\text{age} = 2/3$);
- (b1) $(-1, -1, 1, \dots, 1)$ (a bireflection, $\text{age} = 1$);
- (b2) $(e^{2\pi i/3}, e^{4\pi i/3}, 1, \dots, 1)$ (a complex bireflection, $\text{age} = 1$);
- (b3) $(e^{2\pi i/3}, e^{2\pi i/3}, e^{2\pi i/3}, 1, \dots, 1)$ ($\text{age} = 1$);
- (b4) $(e^{\pi i/2}, e^{3\pi i/2}, 1, 1)$ (a complex bireflection, $\text{age} = 1$);
- (b5) $(e^{\pi i/3}, e^{2\pi i/3}, -1, 1)$ ($\text{age} = 1$);
- (b6) $(e^{\pi i/3}, e^{\pi i/3}, e^{4\pi i/3}, 1)$ ($\text{age} = 1$);
- (b7) $(e^{2\pi i/7}, e^{4\pi i/7}, e^{8\pi i/7}, 1)$ ($\text{age} = 1$);
- (b8) $(e^{\pi i/6}, e^{2\pi i/3}, e^{7\pi i/6}, 1)$ ($\text{age} = 1$).

Additionally, the following spectra also occur for the groups of extraspecial type:

- (c1) $(e^{\pi i/2}, e^{\pi i/2}, 1, 1)$ ($\text{age} = 1/2$).
- (c2) $(e^{\pi i/4}, e^{\pi i/2}, e^{5\pi i/4}, 1)$ ($\text{age} = 1$).
- (c3) $(e^{\pi i/4}, e^{3\pi i/4}, -1, 1)$ ($\text{age} = 1$).
- (c4) $(e^{2\pi i/5}, e^{4\pi i/5}, e^{4\pi i/5}, 1, 1)$ ($\text{age} = 1$).
- (c5) $(e^{2\pi i/5}, e^{2\pi i/5}, e^{6\pi i/5}, 1, 1)$ ($\text{age} = 1$).
- (c6) $(e^{\pi i/2}, e^{\pi i/2}, e^{\pi i/2}, e^{\pi i/2}, 1, 1, 1, 1)$ ($\text{age} = 1$).

4. LINEAR GROUPS GENERATED BY ELEMENTS OF BOUNDED DEVIATION

In this section we will prove Theorem 1.1. The standing hypothesis throughout the section is that

$$(\star\star) : \begin{array}{l} V = \mathbb{C}^d, d > 1, G \text{ is a finite irreducible subgroup of } \mathcal{G} = GL(V), \mathcal{X} \subset \mathcal{G}, \\ C \geq 4 \text{ a given constant, } Z(\mathcal{G})G = \langle \mathcal{X} \rangle, \text{ and } d_2(g)^2 \leq C \text{ for all } g \in \mathcal{X}. \end{array}$$

Let χ denote the character of $Z(\mathcal{G})G$ afforded by V .

4.1. The imprimitive case. Here we consider the case where G (transitively) permutes the m summands of a decomposition $V = V_1 \oplus \dots \oplus V_m$, $m > 1$. For any $g \in \mathcal{X}$, let $\mu(g)$ denote the number of subspaces V_i that are moved by g . Then g fixes (setwise) precisely $m - \mu(g)$ subspaces V_i , whence $|\chi(g)| \leq (m - \mu(g)) \dim(V_1)$. It follows that

$$(7) \quad C \geq d_2(g)^2 = 2(\chi(1) - |\chi(g)|) \geq 2\mu(g) \dim(V_1).$$

Since $Z(\mathcal{G})G = \langle \mathcal{X} \rangle$, there must be some $g \in \mathcal{X}$ which acts nontrivially on the set $\{V_1, \dots, V_m\}$, for which $\mu(g) \geq 2$. Thus $\dim(V_1) \leq C/4$ (also see Example 4.1 for a partial converse).

Now we choose a G -invariant decomposition $V = V_1 \oplus \dots \oplus V_m$ with $m > 1$ smallest possible. This means that the induced action π of G on the set $\{V_1, \dots, V_m\}$ is primitive. Assume in addition that (G, V) does not satisfy the conclusion (iii) of Theorem 1.1, i.e. $\pi(G) \not\cong A_m$. By [LS, Corollary 3], $\mu(g) > 2(\sqrt{m} - 1)$ (for some $g \in \mathcal{X}$). On the other hand, as shown above, $1 \leq k := \dim(V_1) \leq C/4$. Now (7) implies that $m < (1 + C/4k)^2$ and so $\dim(V) < k(1 + C/4k)^2 =: h(k)$. Since the function $h(k)$ is decreasing on $[1, C/4]$, we see that $\dim(V) < h(1) = (1 + C/4)^2$. Thus we have shown that if G is as in $(\star\star)$ and G is imprimitive, then either G satisfies Theorem 1.1(iii), or $\dim(V) < (1 + C/4)^2$. Observe that $(1 + C/4)^2 < f(C)$ as $C \geq 4$. Hence we have proved Theorem 1.1 in the case G is imprimitive.

Example 4.1. Let $C \geq 4$, $V_1 = \mathbb{C}^k$ with $1 \leq k \leq C/4$, and let $H \leq GL(V_1)$ be any subgroup generated by $\{[x^{-1}hx, y] \mid x, y \in H\}$ for a fixed element $h \in H$ (for instance, one can consider any quasisimple subgroup H and any $h \in H \setminus Z(H)$). Then for any $m \geq 2$, the subgroup $G = H \wr S_m < GL_{mk}(\mathbb{C})$ is generated by g^G for some element $g \in G$ satisfying $d_2(g)^2 \leq C$. Indeed, we may write $V = \mathbb{C}^{mk}$ as the set of m -tuples (v_1, \dots, v_m) , $v_i \in V_1$, and define g via $g(v_1, v_2, v_3, \dots, v_m) = (v_2, h(v_1), v_3, \dots, v_m)$; in particular, $\text{Tr}(g) = (m-2)k$ and so $d_2(g)^2 = 2(\dim(V) - |\text{Tr}(g)|) = 4k \leq C$. For any $x, y \in H$, observe that G contains the elements

$$\tilde{x} : (v_1, v_2, \dots, v_m) \mapsto (x(v_1), v_2, \dots, v_m), \quad \tilde{y} : (v_1, v_2, \dots, v_m) \mapsto (y(v_1), v_2, \dots, v_m).$$

Then $K := \langle g^G \rangle$ contains the element $[\tilde{x}^{-1}g^2\tilde{x}, \tilde{y}] = (\tilde{x}^{-1}g\tilde{x})^2 \cdot (\tilde{y}\tilde{x}^{-1} \cdot g \cdot \tilde{x}\tilde{y}^{-1})^{-2}$ which acts on V via $(v_1, v_2, \dots, v_m) \mapsto ([x^{-1}hx, y](v_1), v_2, \dots, v_m)$. The assumption on H now implies that $K > H \times 1 \times \dots \times 1$. But g induces the transposition $(1, 2)$ while acting on the m -tuples (v_1, \dots, v_m) , $v_i \in V_1$. Hence $K > H^m$ and $K/H^m \cong S_m$, and so $K = G$ as stated.

4.2. Tensor decomposable case. Here we assume that G is primitive but tensor decomposable on V : $\chi = \alpha_1 \otimes \dots \otimes \alpha_n$, where $n \geq 2$, $\alpha_i \in \text{Irr}(G)$ is primitive and tensor indecomposable for each i , and $\alpha_1(1) \geq \dots \geq \alpha_n(1) \geq 2$. Then we can find $g \in \mathcal{X}$ such that $d_2(g)^2 \leq C$ and $|\alpha_n(g)| < \alpha_n(1)$. In the case $\beta := \alpha_n$ is tensor induced, among such elements g we can find one that acts nontrivially on the set of tensor factors of β (as otherwise $\beta|_G$ would be tensor decomposable). By Theorem 1.3 applied to (G, β, g) , $\beta(1) - |\beta(g)| \geq \delta := (3 - \sqrt{5})/2$; moreover, if $\beta(1) \geq 13$ then $\beta(1) - |\beta(g)| \geq 2$. In the latter case,

$$C \geq 2(\alpha(1)\beta(1) - |\alpha(g)\beta(g)|) \geq 2\alpha(1)(\beta(1) - |\beta(g)|) \geq 4\alpha(1),$$

and so $\alpha(1) \leq C/4$, where $\alpha := \alpha_1 \otimes \dots \otimes \alpha_{n-1}$. By the choice of α_n , we get $\beta(1) \leq \alpha(1)$ and so $\chi(1) \leq C^2/16 < f(C)$. It remains to consider the case $2 \leq \beta(1) \leq 12$.

Then

$$C \geq 2(\alpha(1)\beta(1) - |\alpha(g)\beta(g)|) \geq 2\alpha(1)(\beta(1) - |\beta(g)|) \geq 2\delta\alpha(1),$$

and so $\alpha(1) \leq C/2\delta$. Therefore,

$$\chi(1) \leq 12 \cdot C/2\delta = 12C/(3 - \sqrt{5}) < 16C < \mathfrak{f}(C),$$

and we are done.

Example 4.2. Given any $C \geq 4$, choose $m = 1 + \lfloor C/4 \rfloor$. Let S_m act on \mathbb{C}^{m-1} as on its deleted natural permutation module. This induces a natural action of $G = S_m \times S_m$ on $V = \mathbb{C}^{m-1} \otimes \mathbb{C}^{m-1}$. Consider the element $g_1 = (\tau, 1)$ and $g_2 = (1, \tau)$, where τ is the transposition $(1, 2) \in S_m$. Then $\chi(1) = (m-1)^2$, $\chi(g_i) = (m-1)(m-3)$ and so $d_2(g_i)^2 = 4(m-1)$. By the choice of m , we see that $d_2(g_i)^2 \leq C$ is very close to C and $\dim(V)$ is very close to $C^2/16$. Clearly, G satisfies the set-up $(\star\star)$ with $\mathcal{X} := g_1^G \cup g_2^G$. (Adding to G an involution inverting the two factors \mathbb{C}^{m-1} of V , we then have $G = \langle g_1^G \rangle$.)

4.3. Tensor induced case. Consider the case G is tensor induced on V : $V = V_1 \otimes V_2 \otimes \dots \otimes V_m$, with $\dim(V_i) = a > 1$ and G permutes the m tensor factors V_1, \dots, V_m (transitively). Then we can find $g \in \mathcal{X}$ such that $d_2(g)^2 \leq C$ and g acts nontrivially on the set $\{V_1, \dots, V_m\}$. By Lemma 2.25, $|\chi(g)| \leq a^{m-1} \leq \chi(1)/2$. Hence $C \geq 2(\chi(1) - |\chi(g)|) \geq \chi(1)$.

4.4. Extraspecial case. Here we consider the case (iii) of [GT3, Proposition 2.8]. In this case, $\dim(V) = p^m$ for some prime p and some integer $m \geq 2$. Pick any $g \in \mathcal{X} \setminus Z(\mathcal{G})$. By [GT1, Lemma 2.4], $|\chi(g)| \leq p^{m-1/2} \leq \chi(1)/\sqrt{2}$. Thus $C \geq 2(\chi(1) - |\chi(g)|) \geq \chi(1)(2 - \sqrt{2})$ and so $\chi(1) \leq C(1 + 1/\sqrt{2})$.

4.5. Almost quasi-simple groups. Here we consider the case G is a finite almost quasi-simple group that satisfies $(\star\star)$. In particular, $L := G^{(\infty)}$ is quasi-simple and $L/Z(L)$ is the unique non-abelian composition factor of G . By the above, we may assume in addition that G is primitive and tensor indecomposable on V . Since $d > 1$ and $Z(\mathcal{G})G = \langle \mathcal{X} \rangle$, there exists $g \in G \setminus Z(G)$ with $d_2(g)^2 \leq C$. By Lemma 2.19 and its proof, L acts irreducibly on V and there is $h \in L \setminus Z(L)$ such that $d_2(h)^2 \leq 4C$.

First assume that $L \not\cong A_n$ for any $n \geq 10$. Then by Proposition 2.22(i) and Lemma 2.23, $|\chi(h)/\chi(1)| \leq 19/20$. It follows that $4C \geq d_2(h)^2 = 2(\chi(1) - |\chi(h)|) \geq \chi(1)/10$, and so $\dim(V) \leq 40C \leq \mathfrak{f}(C)$.

We may now assume that $L = A_n$ for some $n \geq 10$, and moreover $V|_L$ is not isomorphic to the deleted permutation module of L (as otherwise (G, V) satisfies Theorem 1.1(ii)). Up to scalars we may also assume that $A_n \leq G \leq S_n$. By Proposition 2.22(ii), $|\chi(g)/\chi(1)| \leq 1/2 + (n-1)/2n = 1 - 1/2n$, whence

$$(8) \quad C \geq d_2(g)^2 = 2(\chi(1) - |\chi(g)|) \geq \chi(1)/n.$$

In particular, if $n \leq 40$, then $\chi(1) \leq 40C \leq f(C)$. Henceforth we may assume that $n \geq 41$.

Now we choose $\lambda \vdash n$ such that $\chi|_L$ is an irreducible constituent of $\rho^\lambda|_L$, where $\rho^\lambda \in \text{Irr}(\mathbf{S}_n)$ is labeled by λ , and apply Lemma 3.2 to ρ^λ . Assume we are in the former case of Lemma 3.2(ii). Then $\chi(1) \geq \rho^\lambda(1)/2 \geq n(n-1)(n-2)(n-7)/48 > 26n^2$ as $n \geq 41$. Together with (8), we now have that $26n^2 < \chi(1) \leq nC$. It follows that $n \leq C/26$ and so $\chi(1) < C^2/26 < f(C)$.

It therefore remains to consider the case $\lambda \in R_n(2) \cup R_n(3)$; in particular, $\chi_L = \rho^\lambda|_L$. Consider the case $\lambda \in R_n(2)$. By Lemma 3.3 and its proof,

$$C \geq 2(\chi(1) - |\chi(g)|) \geq 2(\chi(1) - |\chi(\mathbf{t})|) \geq 4n - 12 \geq 152,$$

whence $n \leq 3 + C/4$. Thus $\chi(1) \leq (n-1)(n-2)/2 \leq (1 + C/4)(2 + C/4)/2 \leq f(C)$.

Finally, assume that $\lambda \in R_n(3)$. By Lemma 3.3 and its proof,

$$C \geq 2(\chi(1) - |\chi(g)|) \geq 2(\chi(1) - |\chi(\mathbf{t})|) \geq 2(n-2)(n-5) > 68n,$$

whence $n < C/68$. Hence (8) implies that $\chi(1) \leq nC < C^2/68 < f(C)$. Thus we have proved Theorem 1.1 in the case G is almost quasi-simple (and primitive, tensor indecomposable on V). By [GT3, Proposition 2.8], we have therefore completed the proof of Theorem 1.1.

5. LINEAR GROUPS GENERATED BY ELEMENTS OF AGE ≤ 1

In this section we will address the following situation

(♠): $V = \mathbb{C}^d$, $d > 1$, G is a finite irreducible subgroup of $\mathcal{G} = GL(V)$,
 $\mathcal{X} \subset \mathcal{G}$, $Z(\mathcal{G})G = \langle \mathcal{X} \rangle$, and $0 < \text{age}^*(g) \leq 1$ for all $g \in \mathcal{X}$.

By Corollary 2.18, such groups G satisfy the set-up (★★) of §4 with $C := 9.111$. We will denote by χ the character of $Z(\mathcal{G})G$ afforded by V , and frequently refer to intermediate results established in §4.

5.1. Imprimitive groups. First we record the following easy observation:

Lemma 5.1. *Let a finite subgroup $G < GL(V)$ preserve a decomposition $W = V_1 \oplus \dots \oplus V_n$ of a subspace $W \subseteq V$, with $\dim(V_i) = 1$ and G inducing either \mathbf{S}_n or \mathbf{A}_n while permuting the n subspaces V_1, \dots, V_n . Then for any element $g \in G$ with $\text{age}^*(g) \leq 1$, one of the following holds.*

- (i) g acts either trivially, or as a 2-cycle on $\{V_1, \dots, V_n\}$.
- (ii) Some scalar multiple μg of g is a complex bireflection of order 2 or 3 on V , and g acts as a 3-cycle, or a double transposition on $\{V_1, \dots, V_n\}$. Furthermore, $\text{age}^*(g) = 1$.

Proof. Observe that, if h acts as an m -cycle on $\{V_1, \dots, V_m\}$, then it has minimal polynomial $t^m - \alpha$, and m eigenvalues $\beta e^{2\pi i j/m}$, $0 \leq j \leq m-1$, on $V_1 \oplus \dots \oplus V_m$, for

some $\alpha, \beta \in \mathbb{C}^\times$. In particular,

$$(9) \quad \text{age}^*(h|_{V_1 \oplus \dots \oplus V_m}) \geq (m-1)/2.$$

Now assume $g \in G$ has $\text{age}^*(g) \leq 1$ but g does not satisfy (i). By Lemma 2.2(ii), there is $\mu \in S^1$ such that $\text{age}(\mu g) \leq 1$. By Corollary 2.18, $d_2(g)^2 \leq (2.9)\pi$, and so g cannot move more than 4 subspaces V_i by (7). Thus g acts as a 3-cycle, a double transposition, or a 4-cycle on $\{V_1, \dots, V_n\}$. In the third case, $\text{age}^*(g) > 1$ by (9). In the first two cases, (9), Lemma 2.2(iii), and the condition $\text{age}(\mu g) \leq 1$ force μg to act as a complex bireflection of order 3, resp. 2, on W , and trivially on a complement U to W in V . The last claim in (ii) now follows from (9) and the assumption that $\text{age}^*(g) \leq 1$. \square

Lemma 5.2. *Let $G < GL(V)$ be as in (\spadesuit). Assume that G preserves a decomposition $V = V_1 \oplus \dots \oplus V_n$, with $n > 1$ smallest possible. Let π denote the induced permutation action of $Z(\mathcal{G})G$ on $\{V_1, \dots, V_n\}$. Then one of the following holds:*

(i) $\dim(V_i) = 1$, and $(\pi(G), n) = (\mathbf{S}_n, n), (\mathbf{A}_n, n), (ASL_3(2), 8), (SL_3(2), 7), (\mathbf{A}_5, 6), (D_{10}, 5)$. If $g \in \mathcal{X}$ and $\pi(g)$ is not 1 nor a 2-cycle, then g is a complex bireflection of order 2 or 3 and $\text{age}^*(g) = 1$.

(ii) $\dim(V_i) = 2$, $\text{age}^*(g) = 1$ for any $g \in \mathcal{X}$ with $\pi(g) \neq 1$, and the conclusion (iii) of Theorem 1.4 holds.

Proof. Our assumptions imply that $\pi(G)$ is a primitive subgroup of \mathbf{S}_n , and that there must be some $h \in \mathcal{X}$ that moves $\mu(h) \geq 2$ subspaces V_i . By (7), $2\mu(h) \dim(V_1) \leq 9.111$ and so $\mu(h) \dim(V_1) \leq 4$; in particular, $\dim(V_1) = 1$ or 2. Consider the former case: $\dim(V_1) = 1$; in particular, $\mu(t) \leq 4$ for all $t \in \mathcal{X}$. If $\mu(t) = 2$, resp. 3, for some $t \in \mathcal{X}$, then $\pi(G)$ is a primitive permutation group containing a 2-cycle, resp. a 3-cycle, whence $\pi(G) = \mathbf{S}_n$ or \mathbf{A}_n by [W, Theorem 13.3]. Otherwise $\pi(t)$ is a double transposition for all $t \in \mathcal{X}$ with $\pi(t) \neq 1$. Thus $\pi(G)$ is a primitive subgroup of \mathbf{S}_n generated by some double transpositions. Assume in addition that $\pi(G) \not\cong \mathbf{A}_n$. Then $4 > 2(\sqrt{n} - 1)$ by [LS, Corollary 3], i.e. $n \leq 8$, and we arrive at the primitive permutation groups listed in (i). The second claim in (i) follows from Lemma 5.1(ii).

In the latter case, $\mu(h) = 2$, i.e. h is a transposition. This conclusion in fact holds for any $g \in \mathcal{X}$ with $\pi(g) \neq 1$. Thus $\pi(G)$ is a primitive permutation group generated by transpositions, and so $\pi(G) = \mathbf{S}_n$. Let $D := \text{Ker}(\pi)$ and consider any $g \in G \setminus D$ with $\text{age}(g) \leq 1$. Then we may assume that $g : V_1 \leftrightarrow V_2$ and $g(V_j) = V_j$ for all $j \geq 3$. It is not difficult to see that $\text{Spec}(g|_{V_1 \oplus V_2})$ is the union of two cosets of $C_2 = \langle -1 \rangle$ in S^1 . By Lemma 2.20 (with $m = 2$) and Lemma 2.2(iii), the condition $\text{age}(g) \leq 1$ implies that $g^2 = 1_V$, $\text{age}(g) = 1$, g is trivial on each V_j with $j \geq 3$; in particular, g is a bireflection. This argument also shows that $\text{age}^*(t) = 1$ for all $t \in \mathcal{X}$ with $\pi(t) \neq 1$. We will apply this observation to suitable inverse images (in G) of transpositions $(i, i+1)$, $1 \leq i \leq n-1$, of \mathbf{S}_n to show that G is a split extension of D by \mathbf{S}_n . Indeed, denote the element g we have just analyzed by g_1 . For any

$1 \leq i \leq n-1$, a G -conjugate g_i of g_1 will project onto the transposition $(i, i+1)$ and have age = 1. Hence our observation (applied to g_i) yields that

$$g_i : V_i \leftrightarrow V_{i+1}, \quad g_i^2 = 1_V, \quad (g_i)|_{V_j} = 1_{V_j} \text{ for } j \neq i, i+1.$$

Clearly, $(g_i g_j)^2 = 1_V$ if $|i-j| > 1$. Next, if $v \in V_i$ then

$$(g_i g_{i+1})^3 : v \xrightarrow{g_{i+1}} v \xrightarrow{g_i} g_i(v) \xrightarrow{g_{i+1}} g_{i+1}g_i(v) \xrightarrow{g_i} g_{i+1}g_i(v) \xrightarrow{g_{i+1}} g_i(v) \xrightarrow{g_i} v,$$

and similarly for all $v \in V_j$ with $j \neq i$, whence $(g_i g_{i+1})^3 = 1_V$. Thus $H = \langle g_1, \dots, g_{n-1} \rangle$ is a quotient of S_n , and $DH = G = D \cdot S_n$. It follows that $H \cong S_n$. In fact, one can find a basis (u_i, v_i) of each V_i such that H acts via permuting the indices of the u_i 's, resp. of the v_i 's:

$$(10) \quad g_i : u_i \leftrightarrow u_{i+1}, \quad v_i \leftrightarrow v_{i+1}, \quad u_j \mapsto u_j, \quad v_j \mapsto v_j, \text{ for } j \neq i, i+1.$$

□

Lemma 5.3. *Let $G < GL(V)$ be a finite irreducible subgroup. Assume that G preserves a decomposition $V = V_1 \oplus \dots \oplus V_n$, with $\dim(V_i) = 1$ and G inducing either S_n or A_n while permuting the n subspaces V_1, \dots, V_n , and that $n \geq 10$. Then G contains a complex bireflection of order 3.*

Proof. We will represent elements of G by their matrices with respect to a basis (e_1, \dots, e_n) with $V_i = \langle e_i \rangle_{\mathbb{C}}$. Let D be the normal subgroup of G consisting of diagonal elements, so that $S_n \geq G/D \triangleright A := A_n$. Notice that, as an A -module, every chief factor of D is either the trivial module \mathcal{I} , or the heart \mathcal{H} of the natural permutation module, in characteristic p for some prime p . It is well known that $H^2(A, \mathcal{I}) = 0$ if $n \geq 8$ and $p > 2$. Furthermore, $H^2(A, \mathcal{H}) = 0$ if $n \geq 10$ by the main result of [KP]. It follows that $O_{2'}(D) \cdot A$ splits over $O_{2'}(D)$. Since $D = O_2(D) \times O_{2'}(D)$, we may assume that G contains a subgroup $H = O_2(D) \cdot A$.

Now inside H we can find an inverse image g of order 3 of a 3-cycle in A . We may assume that $g = \text{diag} \left(\begin{pmatrix} a & \\ & b \\ & & c \end{pmatrix}, d_1, \dots, d_{n-3} \right)$, where $abc = 1$ and $d_i^3 = 1$. Since $n-3 \geq 4$, we may also assume that $d_4 = d_5$. Next, in H we can find an element $h = \text{diag} \left(\begin{pmatrix} x & \\ y & \end{pmatrix}, z, \begin{pmatrix} u & \\ v & \end{pmatrix}, w_1, \dots, w_{n-5} \right)$. Then $[g, h] = \text{diag} \left(\begin{pmatrix} a' & \\ b' & c' \end{pmatrix}, 1, \dots, 1 \right)$,

where $a'b'c' \neq 0$; in particular, it permutes V_1, V_2, V_3 cyclically. Notice that, $[g, h] \equiv g^{-1} \pmod{O_2(D)}$ and so $[g, h] \in O_2(D) \cdot C_3$, where $C_3 = \langle g \rangle$. Hence, a suitable 2-power t of $[g, h]$ has order 3, acts as a 3-cycle on $\{V_1, V_2, V_3\}$, and fixes every e_i with $i \geq 4$. Clearly, such a t is conjugate to $\text{diag}(e^{2\pi i/3}, e^{4\pi i/3}, 1, \dots, 1)$ and so it is a complex bireflection of order 3. □

Lemma 5.4. *Let $\mathcal{T} = GL_1(\mathbb{C})^n$ be a maximal torus of $\mathcal{G} = GL_n(\mathbb{C})$, so that $N := N_{\mathcal{G}}(\mathcal{T}) = \mathcal{T} \cdot \mathcal{S}_n$, and $n \geq 5$. Assume $D < \mathcal{T}$ is a finite subgroup which is normalized by the subgroup $\mathcal{T} \cdot \mathcal{A}_n$ of N . Then $D \triangleleft N$.*

Proof. 1) Without loss we may assume that D is a p -group for some prime p . If $\exp(D) = q = p^c$, then D is contained in $T := \{x \in \mathcal{T} \mid x^{p^c} = 1\}$. Using the additive notation, we may identify T with the natural permutation $R\mathcal{S}_n$ -module $M := \langle e_1, \dots, e_n \rangle_R$, where $R := \mathbb{Z}/q\mathbb{Z}$. It suffices now to prove that any \mathcal{A}_n -submodule N of M is \mathcal{S}_n -invariant.

2) Assume that $p \nmid n$, with $n \geq 5$ or $(n, p) = (4, 5)$, and consider the module $L := \{\sum_{i=1}^n a_i e_i \mid a_i \in R, \sum_{i=1}^n a_i = 0\}$. Let \mathcal{H} denote the heart of the natural permutation $\mathbb{F}_p \mathcal{S}_n$ -module. The condition on (n, p) implies that \mathcal{H} is irreducible over \mathcal{A}_n . Now observe that the \mathcal{A}_n -module L is uniserial, with \mathcal{H} as the unique composition factor. Hence any \mathcal{A}_n -submodule Y of L is \mathcal{S}_n -invariant. (Indeed, if $t \in \mathcal{S}_n$ then the \mathcal{A}_n -modules Y and tY have same composition length and so $Y = tY$ as L is uniserial.)

3) Consider the case $p \nmid n$. Then $M = A \oplus B$ as \mathcal{S}_n -modules, where $A = \langle \sum_{i=1}^n e_i \rangle_R$, and $B = \{\sum_{i=1}^n a_i e_i \mid a_i \in R, \sum_{i=1}^n a_i = 0\}$. By the result of 2) applied to the \mathcal{S}_n -module $L := B$, any \mathcal{A}_n -submodule Y of B is \mathcal{S}_n -invariant. On the other hand, \mathcal{S}_n acts trivially on A , whence any \mathcal{A}_n -submodule X of A is obviously \mathcal{S}_n -invariant. Now set $X := N \cap A$ and $Y := N \cap B$. Observe that any composition factor of the \mathcal{A}_n -module $N/(X \oplus Y)$ is a common composition factor of $M/A \cong B$ and $M/B \cong A$. Hence $N = X \oplus Y$ and so it is \mathcal{S}_n -invariant.

4) Finally, we assume $p \mid n$ and consider the natural subgroups \mathcal{A}_{n-1} and \mathcal{S}_{n-1} in \mathcal{S}_n , which fix e_1 . Then $M = A \oplus B$ as \mathcal{S}_{n-1} -modules, where $A = \langle e_1, \sum_{i=2}^n e_i \rangle_R$, and $B = \{\sum_{i=2}^n a_i e_i \mid a_i \in R, \sum_{i=2}^n a_i = 0\}$. Then the conclusion of 2) applied to the \mathcal{S}_{n-1} -module $L := B$ implies that any \mathcal{A}_{n-1} -submodule Y of B is \mathcal{S}_{n-1} -invariant. Also, \mathcal{S}_{n-1} acts trivially on A , whence any \mathcal{A}_{n-1} -submodule X of A is obviously \mathcal{S}_{n-1} -invariant. Now set $X := N \cap A$ and $Y := N \cap B$. As in 3), we see that $N = X \oplus Y$, and so it is \mathcal{S}_{n-1} -invariant. Thus N is invariant under $\langle \mathcal{A}_n, \mathcal{S}_{n-1} \rangle = \mathcal{S}_n$. \square

One of the main results of this subsection is the following

Theorem 5.5. *Let $G < \mathcal{G} := GL(V)$ be a finite irreducible subgroup that preserves a decomposition $V = V_1 \oplus \dots \oplus V_n$, with $n > 1$ smallest possible. Assume in addition that $n \geq 3$ and $\dim(V_i) = 1$.*

(i) *Assume G satisfies (\spadesuit) and contains a non-central element g with $0 < \text{age}^*(g) < 1$. Assume in addition that $\pi(G) \geq \mathcal{A}_n$, where π denotes the permutation action of G on $\{V_1, \dots, V_n\}$. Then there is a finite subgroup $Z < Z(\mathcal{G})$ such that ZG contains a complex reflection.*

(ii) *If (G, V) is a basic non-RT pair, then there is a finite subgroup $Z < Z(\mathcal{G})$ and a complex reflection group $H = G(d, 1, n)$ with $d > 1$ (in the notation of [ST]) such that $ZG = ZH$. Conversely, any $G(d, 1, n)$ with $d > 1$ yields a basic non-RT pair.*

Proof. Fix a basis vector e_i for each V_i , and let $D := \text{Ker}(\pi) \triangleleft G$ be consisting of all the elements of G that act diagonally on the basis (e_1, \dots, e_n) .

- 1) Here we show that *if there is an element $g \in G \setminus D$ with $\text{age}(g) < 1$, then either*
 (a) *D contains a non-scalar element h with $\text{age}(h) < 1$, or*
 (b) *λg is a reflection for some $\lambda = e^{-2\pi it}$, with $0 \leq t < 1/2n$ and $\lambda^2 \cdot 1_V \in G$.*

For, by Lemma 5.1, g has the matrix $\text{diag} \left(\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}, c_3, \dots, c_n \right)$ in the given basis, for some $a, b, c_i \in \mathbb{C}^\times$. Then $\text{Spec}(g) = \{\sqrt{ab}, -\sqrt{ab}, c_3, \dots, c_n\}$. Since g has finite order, we may write $c_j = e^{2\pi i r_j}$ with $0 \leq r_j < 1$ for $j > 2$ and $\{\sqrt{ab}, -\sqrt{ab}\} = \{e^{2\pi i r_1}, e^{2\pi i(r_1+1/2)}\}$ with $0 \leq r_1 < 1/2$. By our assumptions, $1 > \text{age}(g) = 1/2 + 2r_1 + \sum_{i=3}^n r_i$, and so $2r_1 + \sum_{i=3}^n r_i < 1/2$. Observe that $g^2 = \text{diag}(ab, ab, c_3^2, \dots, c_n^2)$ and $\text{age}(g^2) \leq 4r_1 + 2 \sum_{i=3}^n r_i < 1$. Now if g^2 is non-scalar, then we can set $h = g^2$. Assume g^2 is scalar; in particular, $ab = c_i^2$ for all $i > 2$ and $c_3^2 \cdot 1_V = g^2 \in G$. Notice that c_3 has finite order in S^1 as $|g|$ is finite. Then $\text{Spec}(c_3^{-1}g) = \{1, -1, 1, \pm 1, \dots, \pm 1\}$. By Lemma 2.21 (with $m = 2$), the condition $\text{age}(g) < 1$ now implies that $\text{Spec}(c_3^{-1}g) = \{1, -1, 1, \dots, 1\}$, and so $c_3^{-1}g$ is a reflection. Finally, $\text{Spec}(g) = \{-c_3, c_3, \dots, c_3\}$ and $\text{age}(g) < 1$, so $c_3 = e^{2\pi it}$ with $0 \leq t < 1/2n$. Thus $\lambda := c_3^{-1}$ has the properties specified in (b).

2) Now we consider the situation of (ii). Then G contains some non-central element g with $\text{age}(g) < 1$ such that $G = \langle g^G \rangle$. Since G is irreducible and $n \geq 3$, $G \neq D$, and so $g \notin D$. Now we can apply the result of 1) to the element g . In the case $D \ni h$ with h non-scalar and $\text{age}(h) < 1$, we would have $G = \langle h^G \rangle \leq D$ (as (G, V) is a basic non-RT pair), a contradiction. Hence λg is a reflection for some λ as specified in (b). Since $G = \langle g^G \rangle$, we see that $ZG = ZH$ for $Z = \langle \lambda \cdot 1_V \rangle < Z(\mathcal{G})$ and $H = \langle (\lambda g)^G \rangle$ is a finite group generated by reflections. Since the c.r.g. H acts imprimitively on V (inducing S_n on $\{V_1, \dots, V_n\}$), by [ST] we must have that $H = G(de, e, n)$ for some positive integers d, e . Assume $e > 1$. Then ZG contains non-central elements $r := \text{diag}(1, \dots, 1, e^{2\pi i/e})$ and $\lambda^{-1}r$, with $\text{age}(r) = 1/e \leq 1/2$ and $\text{age}(\lambda^{-1}r) = 1/e + n/t < 1$. If $\lambda \cdot 1_V \in G$, then $G = ZG$ contains r . Otherwise, G has index 2 in ZG (since $\lambda^2 \cdot 1_V \in G$) and so either r or $\lambda^{-1}r$ belongs to G . In either case, we see that D contains a non-central element $s \in \{r, \lambda^{-1}r\}$ with $\text{age}(s) < 1$ and $\langle s^G \rangle \leq D < G$, a contradiction. So $e = 1$. Also $d > 1$ as otherwise $H = G(1, 1, n) = S_n$ is reducible on V .

Conversely, we show that any c.r.g. $H = G(d, 1, n) = D : S_n$ with $d > 1$ yields a basic non-RT pair. Indeed, since $D < SL(V)$, for any non-central $x \in D$ we have $0 < \text{age}(x) \in \mathbb{Z}$ and so $\text{age}(x) \geq 1$. Now consider any non-central $y \in H$ with $\text{age}(y) < 1$ (such elements exist, for instance, one can take any transposition in S_n). By our observation and by Lemma 5.1, y induces a transposition, say (12), on $\{V_1, \dots, V_n\}$. We need to show that $K := \langle y^H \rangle$ coincides with H . It is clear that $KD = H$. Next, for $\delta := e^{2\pi i/d}$ we have $z := \text{diag}(\delta, 1, \delta^{-1}, 1, \dots, 1) \in D$

and $K \ni yzy^{-1}z^{-1} = \text{diag}(\delta^{-1}, \delta, 1, \dots, 1)$. It is now easy to see that the set of all K -conjugates of $yzy^{-1}z^{-1}$ generates D , and so $K \geq KD = H$.

3) From now on we will assume that *we are in the situation of (i) but there is no finite subgroup $Z < Z(\mathcal{G})$ such that ZG contains a complex reflection*. By Lemma 2.2(ii), there is $\mu \in S^1$ of finite order such that $\text{age}(\mu g) = \text{age}^*(g) < 1$. Replacing G by $\langle \mu \cdot 1_V \rangle \cdot G$ and g by μg , we may (and will) assume that $\text{age}(g) < 1$. By the conclusion of 1), we see that D contains non-central elements h with $\text{age}(h) < 1$. By Lemma 5.4, D is normalized by the monomial subgroup $S \cong S_n$ of $GL(V)$ (that act via permuting the basis vectors e_1, \dots, e_n). In what follows we will freely conjugate elements of D by elements of S .

Let $A = \{x_1 \mid \exists \text{diag}(x_1, \dots) \in D\}$ be the finite subgroup of S^1 consisting of all the first diagonal entries of all the elements in D . Also, let

$$B = \{x_1/x_2 \mid \exists \text{diag}(x_1, x_2, \dots) \in D\},$$

$$C = \{\text{diag}(z_1, \dots, z_n) \mid z_i \in B, \prod_{i=1}^n z_i = 1\}.$$

Observe that $C \leq D$. Indeed, if $x = \text{diag}(x_1, x_2, x_3, \dots, x_n) \in D$, then some S -conjugate of x equals $y = \text{diag}(x_2, x_1, x_3, \dots, x_n) \in D$, and so $D \ni xy^{-1} = \text{diag}(\alpha, \alpha^{-1}, 1, \dots, 1)$ with $\alpha = x_1/x_2 \in B$. Conjugating xy^{-1} suitably, we see that any diagonal matrix with spectrum $\{\alpha, \alpha^{-1}, 1, \dots, 1\}$ (with counting multiplicities and $\alpha \in B$) belongs to D . Now any matrix $\text{diag}(z_1, \dots, z_n)$ with $z_i \in B$ and $\prod_{i=1}^n z_i = 1$, is the product of $n-1$ diagonal matrices $\text{diag}(z_1, z_1^{-1}, 1, \dots, 1)$, $\text{diag}(1, z_1 z_2, (z_1 z_2)^{-1}, 1, \dots, 1)$, \dots , $\text{diag}(1, \dots, 1, z_1 \dots z_{n-1}, (z_1 \dots z_{n-1})^{-1})$, all having spectrum of indicated shape, and so belongs to D .

4) Set $Z_1 = \{z \cdot 1_V \mid z \in A\} < Z(\mathcal{G})$. Claim that $DZ_1 = CZ_1$. Indeed, consider any $x = \text{diag}(x_1, \dots, x_n) \in D$. Conjugating x suitably, we see that $y_i := x_i/x_n \in B$ for $1 \leq i \leq n-1$ and $x_n \in A$. Now express $y := \text{diag}(y_1, y_2, \dots, y_{n-1}, 1) \in DZ_1$ as $y = t_1 t_2 \dots t_{n-1}$, where

$$t_1 = \text{diag}(y_1, y_1^{-1}, 1, \dots, 1), \quad t_2 = \text{diag}(1, y_1 y_2, (y_1 y_2)^{-1}, 1, \dots, 1), \dots, \\ t_{n-2} = \text{diag}(1, \dots, 1, y_1 \dots y_{n-2}, (y_1 \dots y_{n-2})^{-1}, 1), \quad t_{n-1} = \text{diag}(1, \dots, 1, y_1 \dots y_{n-1}, 1).$$

Notice that $t_1, \dots, t_{n-2} \in C$. If $y_1 \dots y_{n-1} \neq 1$, then obviously $t_{n-2} \in DZ_1 < Z_1 G$ is a complex reflection, a contradiction. It follows that $y_1 \dots y_{n-1} = 1$. Now $y = t_1 \dots t_{n-2} \in C$ and $x = x_n y \in CZ_1$ for all $x \in D$, and so $DZ_1 = CZ_1$, as stated.

Let $|B| = b$. Denoting $\epsilon := e^{2\pi i/b}$, we have $v := \text{diag}(\epsilon, \epsilon, \dots, \epsilon, \epsilon^{1-n}) \in C$ and $CZ_1 \ni \epsilon^{-1}v = \text{diag}(1, \dots, 1, \epsilon^{-n})$. In particular, if $\epsilon^n \neq 1$, then $\epsilon^{-1}v$ is a complex reflection, again a contradiction. Hence $\epsilon^n = 1$ and so $b|n$. Now we turn our attention to the non-central element $h \in D$ with $\text{age}(h) < 1$. Since $DZ_1 = CZ_1$, we may write $h = \lambda c$ for some $\lambda \in S^1$ and $c \in C$. Clearly, $\text{age}^*(c) \leq \text{age}(h) < 1$. By Lemma 2.2(ii) and its proof, there is some $\mu \in S^1$, where μ^{-1} is either 1 or one of eigenvalues of c , such that $\text{age}(\mu c) = \text{age}^*(c) < 1$. By the construction of C , $\mu \in B$. Also, $\det(\mu c) = \mu^n \det(c) = 1$ as $b|n$. Now observe that for any element $u \in SL(V)$,

$0 \leq \text{age}(u) \in \mathbb{Z}$. Applying this observation to μc , we see that $\text{age}(\mu c) \in \mathbb{Z}$. Since $0 \leq \text{age}(\mu c) < 1$, we must have that $\text{age}(\mu c) = 0$, and so $\mu c = 1_V$. Thus $h = \lambda c$ is central, a contradiction. \square

Example 5.6. Let $2|n \geq 6$. We exhibit an example of a finite irreducible (imprimitive) subgroup of $GL_n(\mathbb{C})$ which is generated by elements of $\text{age} = 2/3$, but cannot be generated by complex reflections (up to scalars). First consider any $n \geq 5$. Pick a basis (e_1, \dots, e_n) of $V = \mathbb{C}^n$ and consider $G = \langle y_1, x_2, x_3, \dots, x_{n-2}, z_{n-1} \rangle$, where

$$\begin{aligned} y_1 &: e_1 \leftrightarrow e_2, e_j \mapsto e_j \text{ for } 3 \leq j \leq n-1, e_n \mapsto e^{\pi i/3} e_n, \\ x_i &: e_i \leftrightarrow e_{i+1}, e_j \mapsto e_j \text{ for } j \neq i, i+1, n, e_n \mapsto -e_n, \text{ for } 1 \leq i \leq n-2, \\ z_{n-1} &: e_{n-1} \leftrightarrow e_n, e_j \mapsto e_j \text{ for } 1 \leq j \leq n-3, e_{n-2} \mapsto -e_{n-2}. \end{aligned}$$

Also, consider the subgroup $G_n = \langle y_1^3 = x_1, x_2, \dots, x_{n-2}, z_{n-1} \rangle$ of G . Clearly, both G and G_n induce S_n while permuting the 1-spaces $\langle e_1 \rangle, \dots, \langle e_n \rangle$. Next, $y_1^2 = \text{diag}(1, \dots, 1, e^{2\pi i/3})$, and so $(y_1^2)^G$ generates a normal subgroup E of order 3^n of G ; furthermore, $G = E : G_n$. We claim that G_n is an extension of $F = \{\text{diag}(a_1, \dots, a_n) \mid a_i = \pm 1, \prod_{i=1}^n a_i = 1\}$ by S_n . Indeed, it is clear that $G_n < SL(V)$, the normal subgroup F_1 of all diagonal elements of G_n is contained in F , and $G_n/F_1 \simeq S_n$. We will obtain the claim, showing by induction on $n \geq 5$ that $F_1 = F$. When $n = 5$, a direct check using [GAP] shows that $|G_5| = 2^4 \cdot |S_5|$ and so $F_1 = F$. For the induction step, $\langle x_2, \dots, x_{n-2}, z_{n-1} \rangle$ fixes e_1 and plays the role of G_{n-1} while acting on $\langle e_2, \dots, e_n \rangle$. By the induction hypothesis, $G_n \ni f := \text{diag}(1, 1, \dots, 1, -1, -1)$, whence $F_1 = F$.

Next we show that $K := \langle (y_1)^G \rangle$ equals G , and so G is generated by elements of $\text{age} = 2/3$. Clearly, K induce S_n while permuting the 1-spaces $\langle e_1 \rangle, \dots, \langle e_n \rangle$. Also, $K \ni y_1^2$, and so $K > E$. Observe that $f = y_1^3 \cdot z_{n-1} y_1^3 z_{n-1}^{-1} \in K$, whence $K > F$ and so $K = EG_n = G$.

Finally, assuming $2|n$, we show that any complex reflection in $Z(GL(V))G$ is diagonal, and so G cannot be generated by complex reflections (up to scalars). Assume the contrary: there is some $t \in G$ such that $\text{Spec}(t) = \{\gamma, \delta, \dots, \delta\}$ with $\gamma \neq \delta$ and t is not diagonal. Since $\text{age}^*(t) < 1$, by Lemma 5.1 we may assume that $t \equiv x_1 \pmod{EF}$, i.e. $t = x_1 u$, with $u = \text{diag}(u_1, \dots, u_n)$, $u_j = \epsilon^{m_j}$ for some $m_j \in \mathbb{Z}$ and $\epsilon := e^{\pi i/3}$, and $\sum_{j=1}^n m_j \in 2\mathbb{Z}$. Since $\text{Spec}(t) = \{\sqrt{u_1 u_2}, -\sqrt{u_1 u_2}, u_3, \dots, u_{n-1}, -u_n\}$, we must have $-\gamma = \delta = \epsilon^k$ for some $k \in \mathbb{Z}$. Now

$$-1 = (-\delta^n)^3 = (\det(t))^3 = (\det(u))^3 = \left(\prod_{j=1}^n u_j\right)^3 = \epsilon^{3 \sum_{j=1}^n m_j} = 1$$

(since $2|n$), a contradiction.

Finally, we prove an analogue of Theorem 5.5(ii) for $\text{age} \leq 1$:

Theorem 5.7. Let $G < GL(V)$ be a finite imprimitive, irreducible subgroup. Assume G contains non-central elements g with $\text{age}(g) \leq 1$, and $ZG = Z \cdot \langle g^G \rangle$ for any such element g , where $Z := Z(GL(V))$. Then $\dim(V) \leq 8$.

Proof. Assume the contrary: $\dim(V) \geq 9$ for such a group G . Clearly, G satisfies the set-up (\spadesuit). Hence G satisfies one of the conclusions (i) and (ii) of Lemma 5.2.

1) Suppose the conclusion (i) of Lemma 5.2 holds. Since $n = \dim(V) \geq 9$, $\pi(G) \geq A_n$. Let $D = \text{Ker}(\pi)$ be the subgroup of all diagonal elements of G , in a basis (e_1, \dots, e_n) such that $V_i = \langle e_i \rangle$.

First we consider the case $D \not\leq Z$. Then we may assume that $D \ni x = \text{diag}(x_1, \dots, x_n)$ with $x_1 \neq x_2$. Choosing $s \in G$ with $\pi(s) = (123)$, we get

$$G \ni y = [s, x] = \text{diag}(x_3/x_1, x_1/x_2, x_2/x_3, 1, \dots, 1) = \text{diag}(e^{2\pi ia}, e^{2\pi ib}, e^{2\pi ic}, 1, \dots, 1),$$

where $0 \leq a, c < 1$, $0 < b < 1$, and $a + b + c \in \mathbb{Z}$. It follows that either $a + b + c = 1$, in which case $\text{age}(y) = 1$, or $a + b + c = 2$, in which case $\text{age}(y^{-1}) = 1$. In either case, we have found a diagonal non-central element z with $\text{age}(z) = 1$. It is clear that $Z\langle z^G \rangle$ is diagonal and so cannot contain G , a contradiction.

We have shown that $D \leq Z$ and so V yields an irreducible projective representation of degree $n \geq 9$ of S_n or A_n , which is impossible by degree consideration.

2) Now we assume that the conclusion (ii) of Lemma 5.2 holds: $G = D : S_n$, with $n \geq 3$, $D < GL_2(\mathbb{C})^n$ and the action of S_n described in (10) for a fixed basis (u_i, v_i) of each V_i . Let $A = \{x_1 \mid \exists \text{diag}(x_1, \dots) \in D\}$ be the finite subgroup of $GL_2(\mathbb{C})$ afforded by the action of D on V_1 , with respect to the basis (u_1, v_1) . Also, let

$$B = \{x_1 x_2^{-1} \mid \exists \text{diag}(x_1, x_2, \dots) \in D\},$$

$$C = \{\text{diag}(z_1, \dots, z_n) \mid z_i \in B, \prod_{i=1}^n z_i = I\},$$

where I denotes the identity 2×2 -matrix. Note that, by their definition, B and C are *finite sets*. Consider any $a \in B$. Then we can find $x = \text{diag}(x_1, x_2, x_3, \dots, x_n) \in D$ with $a = x_1 x_2^{-1}$, and some conjugate $y = \text{diag}(x_2, x_1, x_3, \dots, x_n) \in D$ of x , whence $D \ni xy^{-1} = \alpha := \text{diag}(a, a^{-1}, I, \dots, I)$. Conjugating α suitably, we see that any matrix $\text{diag}(I, \dots, I, a, a^{-1}, I, \dots, I)$ belongs to D . Similarly, if $b \in B$, then $\beta := \text{diag}(b, b^{-1}, I, \dots, I) \in D$, and $D \ni \alpha\beta = \text{diag}(ab, a^{-1}b^{-1}, I, \dots, I)$. Conjugating the latter element suitably, we see that $D \ni \text{diag}(ab, I, a^{-1}b^{-1}, I, \dots, I)$ and so $ab \in B$. Thus B is closed under multiplication and so it is a group by finiteness. By the above observation applied to ab , $\gamma = \text{diag}(ab, (ab)^{-1}, I, \dots, I)$ belongs to D , and so does $\delta := \gamma^{-1}\alpha\beta = \text{diag}(I, [a, b], I, \dots, I)$. Note that, since $[a, b] \in SL_2(\mathbb{C})$ (and has finite order $|\delta|$), we have either $\text{age}([a, b]) = 1$, or $[a, b] = I$. In the former case, $\text{age}(\delta) = 1$ and $Z\langle \delta^G \rangle \leq ZD \not\leq G$, a contradiction. Hence, $[a, b] = I$ for all $a, b \in B$, i.e. B is an abelian group. This in turn implies that C is a subgroup of D .

Observe that A normalizes B . (Indeed, for any $x_1 \in A$ and $b \in B$, there is some $x = \text{diag}(x_1, x_2, \dots, x_n) \in D$ and $u = \text{diag}(b, I, b^{-1}, I, \dots, I) \in D$. Hence $D \ni xu = \text{diag}(x_1 b, x_2, \dots)$ and so $B \ni x_1 b(x_2)^{-1} = x_1 b x_1^{-1} \cdot x_1 x_2^{-1}$. But $x_1 x_2^{-1} \in B$, hence $x_1 b x_1^{-1} \in B$ as stated.) Since $D \leq A \times A \times \dots \times A$ and $G = D : S_n$, we see that $C \triangleleft G$. In fact, we claim that $[S_n, D] \leq C$. To prove this, let us identify the action of $\sigma \in S_n$ with its action on $\{u_1, \dots, u_n\}$ and on $\{v_1, \dots, v_n\}$, cf. (10). Then

for any $x = \text{diag}(x_1, x_2, \dots, x_n) \in D$ we have $\sigma^{-1}x\sigma x^{-1} = \text{diag}(b_1, \dots, b_n)$, where $b_i = x_{\sigma(i)}x_i^{-1} \in B$. Now choose $\sigma = (j, j+1)$ for $1 \leq j \leq n-1$. Then we get $b_i = I$ for $i \neq j, j+1$, and $b_j b_{j+1} = x_{j+1}x_j^{-1} \cdot x_j x_{j+1}^{-1} = I$. It follows that $\sigma^{-1}x\sigma x^{-1} \in C$, and so $\sigma^{-1}(xC)\sigma = xC$ in D/C for all $\sigma = (j, j+1)$. Consequently, $\sigma^{-1}(xC)\sigma = xC$ in D/C for all $\sigma \in S_n$, as stated.

We have shown that $C \triangleleft G = DS_n$ and $[D, S_n] \leq C$. Hence the subgroup CS_n is normal in G . Recall that $g_1 = (12)$ has $\text{age} = 1$ and $g_1 \in S_n$. By our assumptions, $ZG = Z\langle(g_1)^G\rangle \leq K := ZCS_n$. It follows that $ZG = K$. Let Φ , resp. Φ_1 , denote the representation of ZG on V , resp. of $G_1 := \text{Stab}_{ZG}(V_1)$ on V_1 . Then $G_1 = ZCS_{n-1}$, where S_{n-1} is acting trivially on V_1 . Hence $\Phi_1(G_1) = \Phi_1(ZC) = \mathbb{C}^\times B$ is abelian. But $\dim(V_1) = 2$, so Φ_1 is reducible. Since $\Phi = \text{Ind}_{G_1}^{ZG}(\Phi_1)$, we conclude that ZG is reducible on V , a contradiction. \square

5.2. Extraspecial case. Here, $G \leq N := N_{\mathcal{G}}(E)$ for some p -group E of extraspecial type. By [GT1, Lemma 2.4], either $|\chi(g)| = 0$, or $|\chi(g)|^2 = |C_{E/Z(E)}(g)|$. It follows that $\Delta(g) \geq p^m(1 - 1/\sqrt{p})$. Recall that $C = 9.111$ in the set-up (\spadesuit). Hence $\dim(V) = p^m \leq 9.111(1 + 1/\sqrt{2})$ and so $p^m \leq 13$. Since we are assuming $\dim(V) \geq 4$, we need to consider the following cases.

- $\dim(V) = p \geq 11$. Then $\Delta(g) \geq 11 - \sqrt{11} > 7.68$, and so $\text{age}^*(g) > 1$.
- $\dim(V) = p^m = 9$. Here, $\Delta(g) \geq 9 - 3\sqrt{3}$; moreover, if $|\chi(g)| \leq 3$ then $\Delta(g) \geq 6$. Thus we may assume that $|\chi(g)| = 3\sqrt{3}$. Next, $E = 3_+^{1+4}$, and the character table of $N = Z(\mathcal{G})E : Sp_4(3)$ has been constructed explicitly by T. Breuer. Now one can verify directly that $N/Z(N)$ contains two classes of elements with $|\chi(g)| = 3\sqrt{3}$; any such an element acts on $E/Z(E) = \mathbb{F}_3^4$ as a symplectic transvection. One of these classes has $\text{age}^* = 1$; the other class and all remaining non-central elements in G have $\text{age}^* > 1$.
- $\dim(V) = p^m = 8$. Here, $\Delta(g) \geq 8 - 4\sqrt{2}$ and $E = C_4 * 2_+^{1+6}$. The character table of $N = Z(\mathcal{G})E : Sp_6(2)$ has been constructed explicitly by Breuer. In particular, $\text{Irr}(N)$ contains two, complex-conjugate, characters of degree 8. Hence it suffices to consider one of these two characters and the classes of g with $|\chi(g)| \geq 4$. Now one can verify directly that $N/Z(N)$ contains three conjugacy classes of elements g with $\text{age}^*(g) = 1$; their spectra are listed in items (b1) and (c6) of Remark 3.9. In all other cases, $\text{age}^*(g) > 1$ by Lemmas 2.20 and 2.21.
- $\dim(V) = p^m = 7$. Here, $\Delta(g) \geq 7 - \sqrt{7}$; moreover, if $|\chi(g)| \leq 1$ then $\Delta(g) \geq 6$. Thus we may assume that $|\chi(g)| = \sqrt{7}$. Next, $E = 7_+^{1+2}$, and the character table of $N = Z(\mathcal{G})E : Sp_2(7)$ has been constructed explicitly by Breuer. In particular, $\text{Irr}(N)$ contains seven characters of degree 7, with exactly six being faithful on E , each of which is uniquely determined by its central character. Hence it suffices to consider one of these six characters. Now one can verify directly that in the cases

where $|\chi(g)| = \sqrt{7}$, the smallest arc of S^1 that contains all eigenvalues of g has length $\delta \geq \pi$, and so $\text{age}^*(g) > 1$ by Corollary 2.18.

- $\dim(V) = p^m = 5$. Here, $\Delta(g) \geq 5 - \sqrt{5}$ and $E = 5_+^{1+2}$. The character table of $N = Z(\mathcal{G})E : Sp_2(5)$ has been constructed explicitly by Breuer. In particular, $\text{Irr}(N)$ contains five characters of degree 5, with exactly four being faithful on E , each of which is uniquely determined by its central character. Hence it suffices to consider one of these four characters. Now one can verify directly that $N/Z(N)$ contains three conjugacy classes of elements g with $\text{age}^*(g) = 1$; their spectra are listed in items (b1), (c4), and (c5) of Remark 3.9. In all other cases, $\text{age}^*(g) > 1$ by Corollary 2.18 (with $\delta \geq 6\pi/5$).

- $\dim(V) = p^m = 4$. Here, $\Delta(g) \geq 4 - 2\sqrt{2}$ and $E = C_4 * 2_+^{1+4}$. The character table of $N = Z(\mathcal{G})E \cdot Sp_4(2)$ has been constructed explicitly by Breuer. In particular, $\text{Irr}(N)$ contains two pairs $(\alpha, \bar{\alpha})$ and $(\beta, \bar{\beta})$ of complex-conjugate characters of degree 4; furthermore, β can be obtained from α by tensoring with the sign character of $Sp_4(2) \simeq S_6$. Hence we may assume that $\chi = \alpha$. Now one can verify directly that $N/Z(N)$ contains three conjugacy classes of elements g with $\text{age}^*(g) = 1/2$ and spectra as listed in items (a1), (a4), and (c1) of Remark 3.9. Fixing an isomorphism between $Sp_4(2)$ and S_6 , we may assume that these three classes project onto the classes of (12), resp. (123), (12)(34)(56), in S_6 . $N/Z(N)$ also contains several classes of elements g with $\text{age}^*(g) = 1$ and spectra as listed in items (b1), (b2), (b4), (c2), and (c3) of Remark 3.9. In all other cases, $\text{age}^*(g) > 1$. Now we show that N contains a subgroup M leading to a basic non-RT pair not of reflection type.

Lemma 5.8. *There is a subgroup $M = C_3 \times (C_4 * 2_+^{1+4}) \cdot A_6 < GL(\mathbb{C}^4)$ which gives rise to a basic non-RT pair not of reflection type.*

Proof. Since $M \triangleright E$, M acts irreducibly on $V = \mathbb{C}^4$. Notice that $Z(\mathcal{G})M = Z(\mathcal{G})[N, N]$ has index 2 in $Z(\mathcal{G})N$. By the above analysis, all non-central elements $g \in M$ with $\text{age}^*(g) < 1$ in M are $[N, N]$ -conjugate to an element g with spectrum $(e^{2\pi i/3}, e^{2\pi i/3}, 1, 1)$ which corresponds to the class of (123) in A_6 . In fact one can choose such an element g in $C_3 \times 2A_6 < M$ with $\text{age}(g) < 1$. We have shown that $g^M = \{h \in M \setminus Z(M) \mid \text{age}(h) < 1\}$ and that M contains no complex reflection. It remains to show that $\langle g^M \rangle = M$.

Denote $C := C_3 \times C_4 < Z(\mathcal{G})$, $E := C_4 * 2_+^{1+4}$, $O = EC$, $M_1 := E \cdot A_6 < M = C_3 \times M_1$, and $K := \langle g^M \rangle$. Since $(123)^{A_6}$ generates A_6 , we must have that $KO = M$. Next, since O/C is the unique minimal normal subgroup of M/C , we see that $KC \geq O$ and so $KC = KO = M$. Observe that $[M, M] = [M_1, M_1] = M_1$. (Indeed, it is easy to check that $[M_1, M_1]$ contains $2_+^{1+4} \cdot A_6$ and so it has index at most 2 in M_1 . But M_1 is a normal subgroup of index 2 in $E \cdot S_6$, and one can check that $E \cdot S_6$ has only two linear characters. It follows that $[M_1, M_1] = M_1$.) Now we have $K \geq [K, K] = [KC, KC] = [M, M] = M_1$. Also, M_1 is a perfect subgroup of

$GL(V)$, whence $M_1 < SL(V)$. But $\det(g) = e^{4\pi i/3} \neq 1$, so $M \geq K > M_1$. Since $M/M_1 \cong C_3$, we conclude that $K = M$. \square

We will need the following complement to Theorem 1.3:

Theorem 5.9. *Let $G < GL(V)$ be a finite, irreducible, primitive, tensor indecomposable subgroup and let $g \in G \setminus Z(G)$. If the G -module V is tensor induced, assume in addition that g acts nontrivially on the set of tensor factors of V . Then the following statements holds.*

- (i) *If $\dim(V) = 2$, then $\text{age}^*(g) \geq 1/5$.*
- (ii) *If $\dim(V) = 3$ or 4 , then $\text{age}^*(g) \geq 1/3$.*
- (iii) *If $\dim(V) > 4$, then $\text{age}^*(g) \geq 1/2$.*

Proof. 1) First we consider the case V is tensor induced; in particular, $d := \dim(V) = a^m$ for some integers $a, m \geq 2$, and $\Delta(g) \geq d(1 - 1/a)$ by Lemma 2.25. Now if $d = 4$, then $\Delta(g) \geq 2$ and so $\text{age}^*(g) \geq 4/(2.9\pi) > 0.43$ by Proposition 2.17(iii). If $d \geq 5$, then $\Delta(g) \geq 4$ and so $\text{age}^*(g) \geq 8/(2.9\pi) > 0.86$ again by Proposition 2.17(iii) (in fact, $\text{age}^*(g) > 1$ unless $d = 8$). From now on we may assume that V is not tensor induced.

2) Consider the case $d = 2$ and assume that $\text{age}^*(g) < 1/5$. Then we may write $\text{Spec}(g) = \{1, e^{i\alpha}\}$ with $0 < \alpha < 2\pi/5$. It follows that $|\text{Tr}(g)| = 2\cos(\alpha/2) > 2\cos(\pi/5) = (1 + \sqrt{5})/2$ and so $\Delta(g) < (3 - \sqrt{5})/2$, contradicting Theorem 1.3(i).

Next assume that $d = 3$ and $\text{age}^*(g) < 1/3$. Then we may write $\text{Spec}(g) = \{1, e^{i\alpha}, e^{i\beta}\}$ with $0 \leq \alpha \leq \beta \leq \alpha + \beta < 2\pi/3$; in particular,

$$|\text{Tr}(g)|^2 = 3 + 2\cos(\beta) + 4\cos(\beta/2)\cos(\beta/2 - \alpha).$$

Now $\cos(\beta) > -1/2$ and $\cos(\beta/2) > 1/2$ as $0 \leq \beta < 2\pi/3$. Also, $\cos(\beta/2 - \alpha) > 1/2$, since $-\pi/3 < -\alpha/2 \leq \beta/2 - \alpha \leq \beta/2 < \pi/3$. It follows that $|\text{Tr}(g)|^2 > 3 - 1 + 1 = 3$ and so $\Delta(g) < 3 - \sqrt{3}$, contradicting Theorem 1.3(ii).

3) Now we may assume that $d \geq 4$. If we are in the extraspecial case, then $\text{age}^*(g) \geq 1/2$ by the results of §§5.2. Otherwise, by [GT3, Proposition 2.8] we may apply Theorem 3.8. \square

Note that the lower bounds given in Theorem 5.9 are best possible, cf. the examples of $SL_2(5) < GL_2(\mathbb{C})$, $3_+^{1+2} : SL_2(3) < GL_3(\mathbb{C})$, and Table I for examples in dimensions ≥ 4 .

5.3. Tensor decomposable case.

Lemma 5.10. *In the set-up (\spadesuit), assume that the G -module V is primitive and tensor decomposable. Then $d := \dim(V) \leq 10$.*

Proof. Write $V = V_1 \otimes \dots \otimes V_m$, where V_i are irreducible, primitive, tensor indecomposable G -modules of dimension ≥ 2 , and $m \geq 2$.

1) Consider the case where $\dim(V_i) \geq 3$, say for $i = 1$, and set $W := V_2 \otimes \dots \otimes V_m$. Then we can find $g \in \mathcal{X}$ such that $g|_{V_1}$ is non-scalar. In the case the G -module V_1 is tensor induced, among such elements g we can find one that acts nontrivially on the set of tensor factors of V_1 (as otherwise the G -module V_1 would be tensor decomposable). By Theorem 5.9, $\text{age}^*(g|_{V_1}) \geq 1/3$. By Lemma 2.2(iv), $1 \geq \text{age}^*(g) \geq \dim(W) \cdot \text{age}^*(g|_{V_1})$. It follows that $\dim(W) \leq 3$ and so $m = 2$. Again, we can find $h \in \mathcal{X}$ such that $h|_{V_2}$ is non-scalar. Notice that the G -module V_2 is not tensor induced.

Assume $\dim(V_2) = 3$. Then $\text{age}^*(h|_{V_2}) \geq 1/3$ by Theorem 5.9(ii). Now by Lemma 2.2(iv), $1 \geq \text{age}^*(h) \geq \dim(V_1) \cdot \text{age}^*(h|_{V_2})$. It follows that $\dim(V_1) \leq 3$ and so $d \leq 9$.

Assume $\dim(V_2) = 2$. Then $\text{age}^*(h|_{V_2}) \geq 1/5$ by Theorem 5.9(i). Now by Lemma 2.2(iv), $1 \geq \text{age}^*(h) \geq \dim(V_1) \cdot \text{age}^*(h|_{V_2})$. It follows that $\dim(V_1) \leq 5$ and so $d \leq 10$.

2) Now assume that $\dim(V_i) = 2$ for all i ; in particular, the G -module V_i is not tensor induced. Then we can find $g \in \mathcal{X}$ such that $g|_{V_1}$ is non-scalar. By Theorem 5.9(i), $\text{age}^*(g|_{V_1}) \geq 1/5$. It now follows from Lemma 2.2(iv) that $1 \geq \text{age}^*(g) \geq (d/\dim(V_1)) \cdot \text{age}^*(g|_{V_1})$, $d/\dim(V_1) \leq 5$, and so $d \leq 10$. In fact, $d = 4$ or 8 in this case. \square

The example of $(C_5 \times SL_2(5)) * (C_2 \times SU_4(2))$ acting on $\mathbb{C}^2 \otimes \mathbb{C}^5$ shows that the bound 10 in Lemma 5.10 is best possible.

Lemma 5.11. *In the set-up (\spadesuit), assume that the G -module V is primitive and tensor decomposable and that G contains a non-central element g with $\text{age}^*(g) < 1$. Then $\dim(V) \leq 8$.*

Proof. Assume the contrary: $d := \dim(V) \geq 9$. Notice that $d \leq 10$ by Lemma 5.10. It follows that $d = 9$ or 10 , and $V = A \otimes B$, where A and B are irreducible, primitive, tensor indecomposable, not tensor induced, G -modules of dimension > 1 . Since $g \notin Z(G)$, we may assume that $g|_A$ is not scalar.

Assume $\dim(A) = 3$ (and so $\dim(B) = 3$). Then $\text{age}^*(g|_A) \geq 1/3$ by Theorem 5.9(ii), and so $\text{age}^*(g) \geq \dim(B) \cdot \text{age}^*(g|_A) \geq 1$ by Lemma 2.2(iv), a contradiction. Thus $\dim(A) = 2$ or 5 . Assume $\dim(A) = 2$ (and so $\dim(B) = 5$). Then $\text{age}^*(g|_A) \geq 1/5$ by Theorem 5.9(i). Again by Lemma 2.2(iv), $\text{age}^*(g) \geq \dim(B) \cdot \text{age}^*(g|_A) \geq 1$, a contradiction. Finally, let $\dim(A) = 5$ (and so $\dim(B) = 2$). Then $\text{age}^*(g|_A) \geq 1/2$ by Theorem 5.9(iii), and so $\text{age}^*(g) \geq \dim(B) \cdot \text{age}^*(g|_A) \geq 1$, again a contradiction. \square

The example of $(C_5 \times SL_2(5)) * (C_3 \times Sp_4(3))$ acting on $\mathbb{C}^2 \otimes \mathbb{C}^4$ shows that the bound 8 in Lemma 5.11 is best possible.

Lemma 5.12. *Let $g = A \otimes B$, where $A \in G < GL_m(\mathbb{C})$, $B \in H < GL_n(\mathbb{C})$, with $m \geq 3$ and $n \geq 2$. Assume that G and H are finite primitive irreducible subgroups, and that A, B are non-scalar. Then $\text{age}^*(g) > 1$.*

Proof. 1) Assume the contrary: $\text{age}^*(g) \leq 1$. By a well-known result of Blichfeldt, cf. [D], the smallest arc that contains all eigenvalues of any non-central element in any

finite, primitive, irreducible linear group has length $\geq \pi/3$. Thus $\alpha, \beta \geq 1/6$, where $2\pi\alpha$, resp. $2\pi\beta$, is the length of such smallest arc for A , resp. for B . In particular, $\text{age}^*(A), \text{age}^*(B) \geq 1/6$. On the other hand, $\text{age}^*(A) \leq 1/n \leq 1/2$ and $\text{age}^*(B) \leq 1/m \leq 1/3$ by Lemma 2.2(iv), whence $\alpha \leq 1/2$ and $\beta \leq 1/3$. By Lemma 2.2(ii), we can multiply g by a suitable scalar and assume that $\text{age}(g) \leq 1$. Multiplying B by a suitable $\mu \in S^1$ and A by μ^{-1} , we may assume that $\text{Spec}(B) \ni 1, e^{2\pi i\beta}$, and all other eigenvalues of B belong to the arc $[1, e^{2\pi i\beta}]$ of S^1 . Write $A = \text{diag}(e^{2\pi i\alpha_1}, \dots, e^{2\pi i\alpha_m})$ with $0 \leq \alpha_j < 1$.

2) Here we consider the case $n \geq 3$. Then B has a third eigenvalue $e^{2\pi i\delta}$ with $0 \leq \delta \leq \beta$. For $0 < \gamma \leq 1$, observe that

$$\text{age}(e^{-2\pi i\gamma} B) \geq \begin{cases} 1 + \beta + \delta - 3\gamma \geq 1 + \beta - 2\delta, & 0 < \gamma \leq \delta, \\ 2 + \beta + \delta - 3\gamma \geq 2(1 - \beta) > 1, & \delta < \gamma \leq \beta, \\ \beta + \delta + 3(1 - \gamma) \geq \beta + \delta, & \beta < \gamma \leq 1. \end{cases}$$

We will apply this observation to $\gamma = \gamma_j := 1 - \alpha_j$. Since $1 \geq \text{age}(A \otimes B) = \sum_{j=1}^m \text{age}(e^{-2\pi i\gamma_j} B)$, we see that all γ_j must belong to $(\beta, 1]$, and

$$1 \geq \text{age}(A \otimes B) \geq \sum_{j=1}^m (\beta + \delta + 3(1 - \gamma_j)) \geq m\beta + 3 \sum_{j=1}^m \alpha_j = m\beta + 3\text{age}(A).$$

Recall $m \geq 3$, $\beta \geq 1/6$, and $\text{age}(A) \geq 1/6$. It follows that $m = 3$ and $\text{age}(A) = 1/6$. The last equality however contradicts Theorem 5.9(ii) applied to the element A of G .

3) Now we let $n = 2$. For $0 < \gamma \leq 1$, we have

$$\text{age}(e^{-2\pi i\gamma} B) = \begin{cases} 1 + \beta - 2\gamma \geq 1 - \beta, & 0 < \gamma \leq \beta, \\ 2 + \beta - 2\gamma \geq \beta, & \beta < \gamma \leq 1. \end{cases}$$

We will again apply this observation to $\gamma = \gamma_j := 1 - \alpha_j$ to estimate $\text{age}(A \otimes B) = \sum_{j=1}^m \text{age}(e^{-2\pi i\gamma_j} B)$. If at least one γ_j belongs to $(0, \beta]$, then $\text{age}(g) \geq 1 - \beta + (m - 1)\beta \geq 1 + (m - 2)\beta > 1$, a contradiction. Hence, all γ_j belong to $(\beta, 1]$, and so

$$1 \geq \text{age}(A \otimes B) = \sum_{j=1}^m (\beta + 2(1 - \gamma_j)) = m\beta + 2 \sum_{j=1}^m \alpha_j = m\beta + 2\text{age}(A).$$

Recall that $m \geq 3$, $\text{age}(A) \geq 1/6$, and $\beta = \text{age}(B) \geq 1/5$ by Theorem 5.9(i) applied to the element B of H . It follows that $m \leq (1 - 2/6)/(1/5)$ and so $m = 3$. But in this case, Theorem 5.9(ii) applied to the element A of G yields that $\text{age}(A) \geq 1/3$ and so $\text{age}(A \otimes B) \geq 3/5 + 2/3 > 1$. \square

Corollary 5.13. *Let $G < GL(V)$ be a finite primitive irreducible subgroup. Assume that $\dim(V) \geq 5$ and that the G -module V is tensor decomposable. Then, for any $g \in G$ with $\text{age}^*(g) \leq 1$, $ZG \neq Z\langle g^G \rangle$, where $Z := Z(GL(V))$.*

Proof. Write $V = A \otimes B$ for some G -modules A, B of dimension > 1 . Then G is irreducible and primitive on both A and B . Now consider any $g \in G$ with $\text{age}^*(g) \leq 1$. By Lemma 5.12, g must act scalarly on A or on B , say on A . In this case, $H := \langle g^G \rangle$ also acts scalarly on A and so $ZG \neq ZH$ by irreducibility of G on A . \square

5.4. Tensor induced case.

Proposition 5.14. *In the set-up (\spadesuit), assume that the G -module V is primitive, tensor indecomposable, but tensor induced. Then $\dim(V) = 4$ or 8 . Moreover, if $\dim(V) = 8$, then G cannot be generated by its elements h with $\text{age}^*(h) < 1$ (modulo scalars).*

Proof. 1) By the assumptions, there is a tensor decomposition $V = V_1^{\otimes m}$ with $a := \dim(V_1) > 1$ and $m > 1$ such that $G < GL(V_1)^{\otimes m} : S_m$, and there is some $g \in \mathcal{X}$ such that $0 < \text{age}^*(g) \leq 1$ and g acts nontrivially on the m tensor factors of V . By Lemma 2.25, $4.556 > \Delta(g) = \dim(V) - |\text{Tr}(g)| \geq a^{m-1}(a-1)$. It follows that $a = 2$ and $m = 2$ or 3 .

From now on we assume that $(a, m) = (2, 3)$, i.e. $V = V_1 \otimes V_2 \otimes V_3$ and $\dim(V_i) = 2$. Then g must project onto a 2-cycle of S_3 , as otherwise by Lemma 2.25, $\Delta(g) \geq 8 - 2 = 6$ and so $\text{age}^*(g) > 1$. Without loss we may assume that $g = A \otimes B$, with $A < GL(V_1)^{\otimes 2} : S_2$ permuting the two tensor factors V_1 and V_2 , and $B \in GL(V_3)$, and that g has finite order: $g^N = I_8$ for some integer $N > 1$. (Here we let I_n denote the identity $n \times n$ -matrix.) It follows that $A^N \otimes B^N = I_8 = I_4 \otimes I_2$. By (the first sentence of) the proof of Lemma 2.24, we can multiply A by a suitable $\lambda \in \mathbb{C}^\times$ (and B by λ^{-1}) such that $A^N = I_4$ and $B^N = I_2$.

2) Here we show that $\text{age}^*(A) \geq 1/2$ and $\text{age}^*(g) \geq 1$. By our assumptions, there are some bases (e_1, e_2) of V_1 and (f_1, f_2) of V_2 , and matrices $X, Y \in GL_2(\mathbb{C})$ such that, in the basis $(e_1 \otimes f_1, e_2 \otimes f_1, e_1 \otimes f_2, e_2 \otimes f_2)$ of $V_1 \otimes V_2$, $A = \mathbf{j}(X \otimes Y)$, where $\mathbf{j} : e_i \otimes f_j \mapsto e_j \otimes f_i$. Now direct computation shows that

$$\det(A - tI) = t^4 - \text{Tr}(XY) \cdot t^3 + \text{Tr}(XY) \cdot \det(XY) \cdot t - \det(XY)^2.$$

In particular, writing $\text{Spec}(XY) = \{x, xu^2\}$ for some $x, u \in \mathbb{C}^\times$, we get $\text{Spec}(A) = \{x, xu^2, xu, -xu\}$. Hence $\text{age}^*(A) \geq 1/2$ by Lemma 2.20. Now by Lemma 2.2(iv) we have $\text{age}^*(g) \geq 2\text{age}^*(A) \geq 1$. This lower bound is best possible as $\text{age}(\mathbf{j} \otimes I_2) = 1$ (in fact, $\mathbf{j} \otimes I_2$ acts as a bireflection on V).

3) We have shown that any element of G that acts nontrivially on the set of 3 tensor factors of V has $\text{age}^* \geq 1$. In particular, if $\text{age}^*(h) < 1$ for some $h \in G$, then h belongs to the base subgroup $G \cap GL(V_1) \otimes GL(V_2) \otimes GL(V_3)$. Thus $Z(GL(V)) \cdot \langle h \in G \mid \text{age}^*(h) < 1 \rangle < Z(GL(V))G$. \square

5.5. Proof of Theorem 1.7. Let $d := \dim(V) > 4$ and let $G < GL(V)$ satisfy the hypotheses of the Theorem. If G is imprimitive, then the statement follows from Lemma 5.2 and Theorem 5.5(ii). So we may assume that the G -module V is

primitive. Now by Corollary 5.13, V is tensor indecomposable, and so it cannot be tensor induced by Proposition 5.14. The extraspecial case cannot occur either, by the results of §5.2. Thus G is almost quasi-simple by [GT3, Proposition 2.8], i.e. $S \triangleleft G/Z(G) \leq \text{Aut}(S)$ for some simple non-abelian group S .

We can now apply Theorem 3.8; in particular, either $(d, S) = (n-1, A_n)$ or $d \leq 8$. In the former case, up to scalars, $G = S_n$ (in its action on the deleted natural permutation module) and so a c.r.g. Consider the latter case. If $d = 8$, then $S = \Omega_8^+(2)$, $G/Z(G) = S \cdot 2$, and up to scalars, G is the Weyl group of type E_8 . If $d = 7$, then $G/Z(G) = S = Sp_6(2)$, and up to scalars, G is the Weyl group of type E_7 . Assume $d = 6$. If $S = SU_4(2)$, then $G/Z(G) = S \cdot 2$, and up to scalars, G is the Weyl group of type E_6 . If $S = PSU_4(3)$, then $G/Z(G) = S \cdot 2_2$ in the notation of [Atlas] (the other two involutions in $\text{Out}(S)$ does not preserve the 6-dimensional representation in question of $G^{(\infty)}$), and so G is a c.r.g. modulo scalars. In all these cases, there is only one conjugacy class in $G/Z(G)$ that contain non-central elements g with $\text{age}^*(g) < 1$, and these elements g are scalar multiples of reflections. Finally, if $d = 5$, then $G = SU_4(2) \cdot Z(G)$ and so it is also a c.r.g. modulo scalars. \square

5.6. Proof of Theorem 1.4. Let (G, V) satisfy the hypotheses of the Theorem. If the G -module V is imprimitive, then the statement follows from Lemmas 5.2 and 5.3. (Notice that in case (iii) the transpositions in the subgroup S_n act on V as bireflections.) So we may assume that V is primitive, of dimension ≥ 11 . Hence the extraspecial case cannot occur by the analysis in §5.2. Next, the G -module V cannot be tensor decomposable or tensor induced by Lemma 5.10 and Proposition 5.14. Thus we are in the almost quasi-simple case and can apply Theorem 3.8. Since $\dim(V) \geq 11$, we arrive at the conclusion (i). \square

5.7. Proof of Theorem 1.5. Let (G, V) satisfy the hypotheses of the Theorem. First we consider the case where the G -module V is imprimitive and apply Lemma 5.2. In the case (ii) of Lemma 5.2, we arrive at conclusion (ii) of the Theorem (notice that all elements of G with $\text{age}^* < 1$ are contained in D and so cannot generate G modulo scalars). Suppose we are in the case (i) of Lemma 5.2. Then G satisfies the hypotheses of Theorem 5.5(i), and so we are done. So we may assume that V is primitive and $\dim(V) \geq 9$. Hence the extraspecial case cannot occur by the analysis in §5.2. Next, the G -module V cannot be tensor decomposable or tensor induced by Lemma 5.11 and Proposition 5.14. Thus we are in the almost quasi-simple case and can apply Theorem 3.8. Since $\dim(V) \geq 9$, we must now have that $G = S_{d+1}$ modulo scalars, as stated in (i). \square

Remark 5.15. (a) The group $(C_5 \times SL_2(5)) * (C_3 \times Sp_4(3)) < GL_8(\mathbb{C})$ is generated by its elements of $\text{age} < 1$, but yet does not contain any complex reflection by Lemma 5.10. Thus the bound $d \geq 9$ in Theorem 1.5 is best possible.

(b) The case (ii) of Theorem 1.5 indeed occurs, as shown in the following example. Consider the subgroup $A = C_7 \times SL_2(5)$ of $GL_2(\mathbb{C})$ and let G be the wreath product $A \wr S_n$ acting on $V = \mathbb{C}^{2n}$ for any $n \geq 2$. It is easy to check that G is generated by its (non-central) elements with $\mathbf{age} \leq 1$, and G contains non-central elements with $\mathbf{age} = 2/7$. However, G does not contain any complex reflection. For, suppose $g \in G$ is conjugate to $\text{diag}(\alpha, \alpha, \dots, \alpha, \beta)$ for some $\alpha \neq \beta \in S^1$. By Lemma 5.2, $g = \text{diag}(a_1, \dots, a_n) \in A^n$ since $0 < \mathbf{age}^*(g) < 1$. It follows that $\alpha^{14} = 1$ and that some element $x \in SL_2(5)$ has eigenvalues $\mu\alpha, \mu\beta$ for some $\mu \in S^1$ with $\mu^7 = 1$. Now $(\mu\alpha)^{14} = 1$, and so x must be scalar, whence $\alpha = \beta$, a contradiction.

5.8. Proof of Corollary 1.2. By the assumption, $2\pi \cdot \|g\| \leq L$ for some $1 \neq g \in G$. Let N be the (normal) subgroup generated by all elements in G with this property. By Corollary 2.12, N is generated by a set of elements g with $d_2(g)^2 \leq C$, where $C = \max\{4, L^2\}$. Also by [GT3, Lemma 2.5], either $N \leq Z(G)$, or N is irreducible on V . In the former case, the cyclic group $Z(G)$ contains an element $g = e^{2\pi i j/s} \cdot 1_V$ with $s := |Z(G)|$, $1 \leq j \leq s-1$, and $L \geq 2\pi \cdot \|g\| \geq 2\pi \sqrt{\dim(V)}/s$, whence $\dim(V) \leq (Ls/2\pi)^2$. In the latter case, we may apply Theorem 1.1 to N . Assume that the conclusion (i) of Theorem 1.1 does not hold; in particular, $d := \dim(V) > 40C \geq 160$. In the case the conclusion (ii) of Theorem 1.1 holds for N , we have that $G \triangleright M := N^{(\infty)} \cong A_{d+1}$ and M acts irreducibly on V . By Schur's Lemma, $C_G(M) = Z(G)$, and $G/C_G(M) \leq \text{Aut}(M) = M \cdot 2$, whence the conclusion (ii) of Theorem 1.1 holds for G .

Finally, assume that the conclusion (iii) of Theorem 1.1 holds for N , and let D be the normal subgroup of N that fixes each V_i (setwise); in particular, $A_m \leq N/D \leq S_m$. In this case, $40C < d = m \dim(V_1) \leq mC/4$, whence $m > 160$. If, in addition, $m \leq C/4 + 1$, then $C \geq 640$ and so $d \leq mC/4 \leq (C/4 + 1) \cdot C/4 < 4C^2/63 \leq f(C)$. Hence we may assume $m > \max\{160, C/4 + 1\}$. Set $e := \dim(V_1)$ and let \mathcal{C} be the collection of all finite simple groups S with the property that either S is cyclic, or $S \cong X/Y$ for some finite subgroups $Y \triangleleft X < PGL_e(\mathbb{C})$. Observe that every composition factor of D belongs to \mathcal{C} . (Indeed, consider the chain $D = D_0 \triangleright D_1 \triangleright D_2 \triangleright \dots \triangleright D_m = 1$, where D_i is the kernel of the action of D_{i-1} on V_i for $1 \leq i \leq m$. Now let S be any non-abelian composition factor of D_i/D_{i-1} . Thus $S \cong A/B$ for some $B \triangleleft A < GL(V_i)$ since $D_i/D_{i-1} \hookrightarrow GL(V_i)$. Since S is non-abelian, S is also a composition factor of AZ/BZ for $Z := Z(GL(V_i))$. It follows that S is a composition factor of $AZ/Z < PGL(V_i) \cong PGL_e(\mathbb{C})$, i.e. $S \in \mathcal{C}$.) Let R be the largest normal subgroup of N with every composition factor belonging to \mathcal{C} , cf. Lemma 2.26. Then $D \triangleleft R \triangleleft N$. Assume that $R > D$. Then A_m is a composition factor of R and so $A_m \in \mathcal{C}$. The latter inclusion means that $A_m \cong X/Y$ for some finite subgroups $Y \triangleleft X < PGL_e(\mathbb{C})$, with $e \leq C/4 < m-1$ and $m > 160$. This however contradicts the Feit-Tits Theorem, cf. [KIL, Theorem 3]. Thus $R = D$. By Lemma 2.26, $R \triangleleft G$. We have shown that $D \triangleleft G$. Since D is reducible on V , by [GT3, Lemma

2.5] we must have $D \leq Z(G) \cap N = Z(N)$. It follows that V yields an irreducible projective representation of degree $me \leq mC/4 < m^2$ of $N/D \in \{A_m, S_m\}$. Recall that $m > 160$. Using the information on the small degrees of irreducible projective representations of N/D as given in [Ra] and [KT], we see that $me = m(m-3)/2$, $M = N^{(\infty)} \cong A_m$, and $V|_M$ equals the restriction of the Specht module $S^{(m-2,2)}$ (labeled by the partition $(m-2, 2)$) to A_m . But the latter restriction is primitive, whereas V is imprimitive, a contradiction. \square

5.9. Proof of Corollary 1.6. By the assumption and the Reid-Tai criterion [R1], $\text{age}(g) \leq 1$ for some $1 \neq g \in G$. Let N be the (normal) subgroup generated by all elements in G with this property. By [GT3, Lemma 2.5], either $N \leq Z(G)$, or N is irreducible on $V = \mathbb{C}^d$. In the former case, the cyclic group $Z(G)$ contains an element $g = e^{2\pi i j/s} \cdot 1_V$ with $s := |Z(G)|$, $1 \leq j \leq s-1$, and $1 \geq \text{age}(g) \geq \dim(V)/s$, whence $\dim(V) \leq s$. In the latter case, we may apply Theorem 1.4 to N . Assume that the conclusion (i) of Theorem 1.1 holds for N . Then $G \triangleright M := N^{(\infty)} \cong A_{d+1}$ and M acts irreducibly on V . By Schur's Lemma, $C_G(M) = Z(G)$, and $G/C_G(M) \leq \text{Aut}(M) = M \cdot 2$, whence the conclusion (i) of Corollary 1.6 holds for G .

Next, assume that either the conclusion (ii) or (iii) of Theorem 1.4 holds for N . In the case of (ii), define D to be the normal subgroup of N that fixes each V_i (setwise); in the case of (iii), consider the normal subgroup D defined therein. In particular, $A_n \leq N/D \leq S_n$. Also, let \mathcal{C} be the collection of all finite simple groups S with the property that either S is cyclic, or $S \cong X/Y$ for some finite subgroups $Y \triangleleft X < PGL_2(\mathbb{C})$ (and so $S \cong A_5$, as easily seen). Notice $n \geq 6$ since $n \geq d/2$. Arguing as in the last part of the proof of Corollary 1.2, we see that D is the largest normal subgroup of N with all composition factors belonging to \mathcal{C} , and so $D \triangleleft G$ by Lemma 2.26. Since D is reducible on V , by [GT3, Lemma 2.5] we must have that $D \leq Z(G) \cap N = Z(N)$. Now in the case of the conclusion (iii) of Theorem 1.4 for N , we have $N = S_n Z(N)$, with S_n acting reducibly on V , a contradiction. Thus we are in the case of the conclusion (ii) of Theorem 1.4, in particular $n \geq 11$, and V yields an irreducible projective representation of degree n of $N/D \in \{A_n, S_n\}$, again a contradiction. \square

5.10. Small dimension case.

Proposition 5.16. *Let G satisfy the set-up (\spadesuit), with $4 \leq d := \dim(V) \leq 10$. Then one of the following statements holds.*

- (i) G preserves a decomposition $V = V_1 \oplus \dots \oplus V_d$ of V into 1-spaces, and $(\pi(G), d) = (S_d, d), (A_d, d), (ASL_3(2), 8), (SL_3(2), 7), (A_5, 6), (D_{10}, 5)$, if π denotes the induced permutation action of G on $\{V_1, \dots, V_d\}$.
- (ii) $2|d$, and $G = D : S_{d/2} < GL_2(\mathbb{C}) \wr S_{d/2}$, a split extension of $D < GL_2(\mathbb{C})^{d/2}$ by $S_{d/2}$.
- (iii) G preserves a decomposition $V = A \otimes B$, with $\dim(A), \dim(B) > 1$.

- (iv) G preserves a tensor structure $V = A^{\otimes m}$, with $\dim(A) = 2$ and $m = 2, 3$.
- (v) $4 \leq \dim(V) = p^a \leq 9$ for some prime p , and G normalizes a p -group E of extraspecial type, with $|E/Z(E)| = p^{2a}$.
- (vi) G is almost quasi-simple, and G satisfies the conclusions of Theorem 3.8.

Proof. If the G -module V is imprimitive, then the statement follows from Lemma 5.2. Assume V is primitive. Next, (iii), resp. (iv), (v), corresponds to the case when the G -module V is tensor decomposable, resp. tensor induced, or G is in the extraspecial case. Otherwise, by [GT3, Proposition 2.8] G satisfies the hypothesis, and so the conclusions, of Theorem 3.8. \square

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